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Neutrino Mass Generation

in Conformally Invariant Theories

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Erzeugung von Neutrinomassen in Konform Invarianten Theorien:

Das Ziel dieser Arbeit ist es einen systematischen Überblick zu geben, wie Neutrinomassen in konform invarianten Theorien erzeugt werden können. Dabei werden bekannte Szenarien zur Erzeugung von Neutrinomassen wiederholt und hiervon konform invariante Theorien abgeleitet, wobei Unterschiede im notwendigen Teilcheninhalt und in den physikalischen Auswirkungen hervorgehoben werden. In diesem Zusammenhang werden sowohl Regeln, welche bei der Konstruktion von Neutrinomassen in konformen Theorien berücksichtigt werden müssen, als auch topologische Möglichkeiten und Unmöglichkeiten, unter relativ allgemeinen Annahmen, ausgearbeitet.
Die Struktur dieser Arbeit ist bestimmt durch die Massenmatrix für Neutrinos $\mathcal{M}$, welche die Eigenschaften eines Mechanismus zur Neutrinomassenerzeugung zusammenfasst. Die Modelle werden durch ihren Einfluss auf die Massenmatrix und durch ihren Teilcheninhalt unterschieden. Da zusätzliche Symmetrien die allgemeine Form und da Tree-Level und radiative Szenarien die Skala der Einträge von $\mathcal{M}$ unterschiedlich beeinflussen, spielen beide Kriterien eine wichtige Rolle bei der Unterscheidung von Modellen.
Für ein einfaches konformes Modell mit sterilen Neutrinos wird eine phänomenologische Untersuchung durchgeführt, welche die zulässigen Bereiche im Parameterraum der Kopllungskonstanten darstellt.

Neutrino Mass Generation in Conformally Invariant Theories:

The aim of this thesis is to give a systematic overview of how neutrino masses can be generated in conformally invariant theories. This is done by revisiting well-known scenarios of neutrino mass generation and deducing conformally invariant realizations, where differences in the necessary particle content and the physical implications are highlighted. In this context rules that have to be obeyed when building neutrino masses in conformal theories as well as topological possibilities and impossibilities under relatively general assumptions are elaborated.
In this work we systematically analyze the neutrino mass matrix $\mathcal{M}$, which summarizes the properties of a mass generation mechanism. Models are distinguished by their implications on the mass matrix and by their particle content. As additional symmetries have influence on the general form of $\mathcal{M}$ and tree-level and radiative scenarios affect the scale of its entries differently, both criteria play a crucial role in the distinction of models.
For a simple conformal model containing sterile neutrinos a phenomenological study is performed showing the viable areas in parameter space of the coupling constants.

Contents

Chapter 1

Introduction

It is a final tribute to the success of the Standard Model of Particle Physics (SM) that in 2013 the Nobel Prize in physics has been awarded to Peter Higgs and François Englert for the discovery of a mechanism of spontaneous symmetry breaking. Even the last particle predicted by the SM, the Higgs particle, which is related to this mechanism responsible for the masses of all elementary particles, has been found in the year 2012 at CERN's Large Hadron Collider (LHC). Within the last decades the SM has proven to be capable of describing almost all phenomena in particle physics upto energy scales that have been tested on earth with very high precision.

At the same time the discovery of the Higgs particle concludes one chapter in the history of physics to finally open the door to the era of Beyond the Standard Model physics. For though the SM has been successful we know that it is not complete.

First of all the SM only describes three elementary forces. There is still no satisfying quantum theory for gravitation and thus at the Planck scale, where the gravitational force becomes important, the SM has to break down. Secondly oscillation experiments give rise to small neutrino masses, which are, due to the absence of right-handed neutrinos in the SM, not immediately implicated. Furthermore, there are more aesthetical shortcomings like the Hierarchy Problem, which refers to the Higgs mass not being stable towards radiative corrections. Beyond that problems like the Strong CP Problem or the unknown nature of Dark Matter are still unsolved. The possibility of simultaneously solving the Hierarchy Problem and generating neutrino masses will be discussed in the course of this thesis.

To provide an alternative solution to the Hierarchy Problem besides Supersymmetry, W. A. Bardeen suggested that conformal invariance could solve this problem. He argued that the quadratic divergences arising in radiative corrections to the Higgs mass are a relict of the choice of the regularization procedure only and can thus not be considered as a naturalness problem. Once accepting this argument and introducing conformal

invariance as a fundamental property of the classical Lagrangian two questions arise. The first and more general question is how conformally invariant theories can bear any mass scale at all, especially in a way that the masses of the SM particles are reproduced. The second question is how small neutrino masses can be generated in a natural way. The first question can partly be answered by the work done by S. Coleman and E. Weinberg who suggested a mechanism of radiative symmetry breaking [1]. Unfortunately simply making the SM conformally invariant by dismissing the negative mass term in the Higgs potential and applying the Coleman-Weinberg (CW) mechanism does not yield the right Higgs and top quark masses. The second question, however, might be immediately connected to this problem. H. Nicolai and K.A. Meissner for example suggested an extension of the Standard Model by three right-handed neutrinos and a singlet scalar [2]. In this way they showed that they could generate the right Higgs and top quark masses and at the same time small neutrino masses.
Motivated by those findings this work will give a systematic overview of how neutrino masses can be generated in conformally invariant theories. Well-known mechanisms for the generation of neutrino masses in non-conformal theories will be reviewed and compared to known and new conformally invariant theories yielding neutrino masses. Different theories will be organized by their underlying symmetries, their particle content and their phenomenology.
To have a first criterion for the phenomenological viability and the organization of several theories, a phenomenological study for the introduction of right-handed sterile neutrinos in a conformally invariant way will be performed.

The thesis is structured as follows. The second chapter deals with the theory of scale and conformal transformations and how conformal invariance can solve the Hierarchy Problem. Furthermore it will be shown how the effective potential can be calculated and the symmetry of a theory can be broken radiatively according to the CW mechanism.
In chapter 3 the physics of neutrinos will be discussed. Phenomenological properties deduced from oscillation experiments and the distinction between Dirac and Majorana neutrinos will be outlined. Beyond that useful technicalities and a first example of neutrino mass generation will be presented.
In chapter 4 a conformally invariant theory containing sterile neutrinos, the Meissner-Nicolai model, will be phenomenologically analysed by scanning the parameter space of Dirac and Majorana Yukawa coupling constants showing phenomenological viable areas in a coloured 2D map.
Chapter 5 collects conformally invariant models for the generation of neutrino masses. In the first section, having the one-flavour 2×2 mass matrix as a guiding principle, conformally invariant theories will be investigated that affect the diagonal entries of the mass matrix.

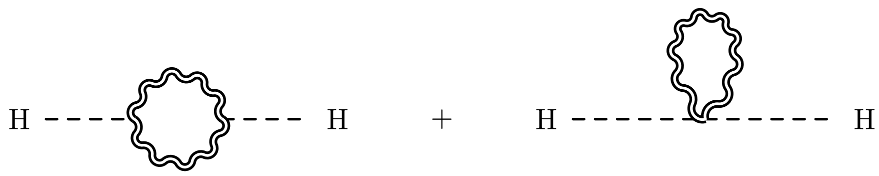
In the second section an additional $U(1)$ symmetry will be introduced that separates a Hidden Sector (HS) from the Standard Model. Within this symmetry a further sterile neutrino with HS charge is introduced that extends the one-flavour mass matrix to 3×3. Different models will be investigated that influence the diagonal entries of the mass matrix.

Chapter 2

Scale and Conformal Symmetry

2.1 The Hierarchy Problem

One of the most important shortcomings of the Standard Model of Particle Physics (SM) is the Hierarchy Problem. It arises when considering the SM as an effective theory assuming that it is valid upto a characteristic scale Λ and taking this scale as the physical cut-off of the theory. Thus all physics below this scale is determined by the well-known rules of quantum field theory and all loop integrals have to be evaluated upto the momentum cut-off Λ. This scale of new physics may lie somewhere between the electroweak scale ~ 100 GeV and the Planck scale $\sim 10^{19}$ GeV. Motivated by the observation that all three couplings strengths, i.e. of QCD, the weak interaction and hypercharge, meet approximately in one point at around 10^{15} GeV, this value is often taken as the scale where new physics has to be introduced unifying all three forces in a so called Grand Unified Theory (GUT) [3]. Calculating now the radiative corrections to the Higgs mass m_H we have to take into account the following diagrams:



where the dashed line represents the Higgs particle and the double line in the loops represents any particle that couples to the Higgs particle in the corresponding way. From that we can find the radiative corrections to be quadratic in the cut-off and the relation between bare mass and physical mass is given by

$$m_H^2 = m_0^2 + Cg^2\Lambda^2, \tag{2.1}$$

where m_0 is the bare mass, g represents a corresponding coupling constant and C is a constant. For a pragmatist this does not yield a problem in the first place as within the idea of renormalization we can only gain knowledge from the observable Higgs mass m_H, whereas the bare mass m_0 is a free parameter of the theory and can be chosen such that eq. (2.1) yields the right mass. It is thus not a technical shortcoming but rather a matter of aesthetics when considering the degree of necessary fine-tuning of m_0. The physical Higgs mass is of electroweak scale, i.e. ~ 100 GeV, while choosing Λ to be of GUT scale yields $\Lambda \sim 10^{15}$ GeV. This means that m_0 has to be chosen such that

$$m_0^2 \propto -10^{30}\text{GeV}^2 + 10^4\text{GeV}^2. \tag{2.2}$$

As we can see the bare mass has to be chosen of GUT scale as well, however, has to be fine-tuned such that after radiative corrections a small rest, namely the Higgs mass, remains. Thus the degree of fine-tuning necessary to achieve that can be expressed by the ratio

$$\frac{10^4\text{GeV}^2}{10^{30}\text{GeV}^2} = 10^{-26}. \tag{2.3}$$

The huge degree of fine-tuning necessary for the Higgs mass to be of the right scale in the SM is referred to as the Hierarchy Problem.[1]

It generally occurs for scalar corrections which are not protected by a symmetry. The same does not appear for fermionic corrections as those are protected by the approximate chiral symmetry at high energies. This is because exact chiral symmetry would forbid mass terms for fermions at tree-level as well as their radiative generation. For high energies masses can be neglected and chiral symmetry can approximately be restored. In this way chiral symmetry, even if not an exact symmetry, only allows fermion masses to be logarithmically divergent.

In the same way supersymmetric models avoid scalar quadratic divergences by introducing a boson-fermion symmetry [4]. The quadratic divergence of the Higgs particle is cancelled by its supersymmetric fermionic partner, assuming that the cut-off for fermions and bosons is chosen alike.

Beyond other attempts, like the introduction of extra dimensions resulting in a shift of the Planck scale down to the electroweak scale (see e.g. [5]), conformal symmetry can be argued to avoid the Hierarchy Problem as discussed in the following.

[1] Strictly speaking the SM for itself does not have a Hierarchy Problem. It is rather the assumption of new physics at a higher scale that causes the problem.

2.2 Conformal Symmetry and Related Issues

2.2.1 Symmetries in Classical and Quantum Field Theories

In classical field theories the dynamics of the fields are determined by the variational principle, i.e. by the variation of the action given by [6]

$$\delta S = \delta \int \mathrm{d}^4x \, \mathscr{L} = 0 \,, \tag{2.4}$$

where $\mathscr{L} = \mathscr{L}(\boldsymbol{\phi}, \partial_\mu \boldsymbol{\phi})$ is the Lagrangian in dependence of the collection of fields $\boldsymbol{\phi} = (\phi_1, \phi_2, ...)$ and their derivatives, which includes all kinds of fields, not only scalars. The dependence of the fields on the space-time vector x was and will be suppressed in the following unless it is necessary for the sake of comprehensibility.
If now under the transformation of the fields, given by

$$\boldsymbol{\phi} \to \boldsymbol{\phi} + \delta\boldsymbol{\phi} \tag{2.5}$$

the action transforms like

$$\delta S = \int \mathrm{d}^4x \, \Delta \,, \tag{2.6}$$

then the current

$$j^\mu = \frac{\partial \mathscr{L}}{\partial(\partial_\mu \boldsymbol{\phi})} \delta\boldsymbol{\phi} \tag{2.7}$$

fulfils the equation

$$\partial_\mu j^\mu = \Delta. \tag{2.8}$$

If $\Delta = 0$ we say that classically the theory has a symmetry and the current j^μ is conserved.[2]
In a quantum field theory dynamics are not any more determined by the variational principle. Even the objects of interest are not the fields themselves but rather their expectation values. The central object that determines the physics of a quantum field theory is the partition function

$$Z(J) = \int \mathscr{D}\boldsymbol{\phi} \, e^{i[S + \int \mathrm{d}^4y \, J_a \phi_a]} \,, \tag{2.9}$$

where the right-hand side of the equation is a path integral over all fields ϕ_a and J_a are external fields.

[2]We also say that the theory has a symmetry if Δ is a total four divergence $\partial_\mu f^\mu$ for arbitrary f^μ. We can then redefine the current as $j^{\mu\prime} = j^\mu - f^\mu$ such that we get the conservation law $\partial_\mu j^{\mu\prime} = 0$.

Under a transformation like in eq. (2.5) we find for the current of eq. (2.7) the relation

$$\begin{aligned}\partial_\mu\langle 0|Tj^\mu(x)\phi_{a_1}(x_1)...\phi_{a_n}(x_n)|0\rangle &= \langle 0|\Delta(x)\phi_{a_1}(x_1)...\phi_{a_n}(x_n)|0\rangle \\ &-\sum_{j=1}^{n}\langle 0|T\phi_{a_1}(x_1)...\delta\phi_{a_j}(x)\delta^4(x-x_j)...\phi_{a_n}(x_n)|0\rangle\,,\end{aligned}\tag{2.10}$$

which is called the Ward-Takahashi identity. If $\Delta = 0$ the Ward-Takahashi identity tells us that the current j_μ is conserved in a correlation function up to contact terms.
Note that this statement is true only under the assumption that the measure $\mathcal{D}\boldsymbol{\phi}$ is invariant under the given transformation of the fields. If it is not the Ward-Takahashi identity is not fulfilled an we say that the symmetry is anomalous.

2.2.2 Scale Transformations

Instead of directly defining conformal transformations and how they apply to fields we want to start with scale transformations which are more intuitive and easier to handle. It is the next section where we will properly define conformal transformations and relate those to scale transformations within the conformal group. In this context we will also find the conditions under which conformal invariance leads to scale invariance and conditions under which scale invariance leads to conformal invariance. For a more detailed discussion of scale and conformal transformations see ref. [7] and [8].
Scale transformations are space-time transformations defined by

$$\alpha : x \to e^\alpha x\,,\tag{2.11}$$

where α is a real number. The corresponding transformations of the fields are given by

$$\alpha : \boldsymbol{\phi}(x) \to e^{\alpha d}\boldsymbol{\phi}(e^\alpha x)\,,\tag{2.12}$$

where the different fields were summarized in the vector $\boldsymbol{\phi}$ like before and d is a matrix in this field space, which is diagonal and carries the corresponding mass dimensions as diagonal entries. In 4 dimensions this is one for bosons and 3/2 for fermions.
The infinitesimal form of this is thus given by[3]

$$\delta\boldsymbol{\phi} = (d + x^\mu\partial_\mu)\boldsymbol{\phi}\,,\tag{2.13}$$

[3]From eq. (2.11) we find $\delta x = (e^\alpha - 1)x \approx \alpha x$.
Using then eq. (2.12) we get $\delta\phi = (\partial_\mu\phi)\delta x^\mu + \Delta\phi = \alpha x^\mu\partial_\mu\phi + e^{\alpha d}\phi - \phi \approx \alpha(d + x^\mu\partial_\mu)\phi$.

where the global factor α was absorbed in a redefinition of $\delta\boldsymbol{\phi}$ but has to be restored when necessary. We thus say that a theory is scale invariant if it is the total divergence

$$\delta\mathscr{L} = 4\mathscr{L} + x^\mu \partial_\mu \mathscr{L} = \partial_\mu \left(x^\mu \mathscr{L} \right). \tag{2.14}$$

According to eq. (2.6) Δ is given by $\Delta = \partial_\mu \left(x^\mu \mathscr{L} \right)$. Thus we redefine the current j_μ such that for $J^\mu = j^\mu - x^\mu \mathscr{L}$ we find the conservation law $\partial_\mu J^\mu = 0$. This current is explicitly given by

$$\begin{aligned} J^\mu &= \boldsymbol{\Pi}^\mu \delta\boldsymbol{\phi} - x^\mu \mathscr{L} \\ &= \boldsymbol{\Pi}^\mu d\boldsymbol{\phi} + \boldsymbol{\Pi}^\mu x^\lambda \partial_\lambda \boldsymbol{\phi} - x^\mu \mathscr{L} \\ &= \boldsymbol{\Pi}^\mu d\boldsymbol{\phi} + x_\lambda T^{\mu\lambda}, \end{aligned} \tag{2.15}$$

where $T^{\mu\lambda}$ is the canonical energy-momentum tensor given by

$$T^{\mu\lambda} = \boldsymbol{\Pi}^\mu \partial^\lambda \boldsymbol{\phi} - g^{\mu\lambda} \mathscr{L} \tag{2.16}$$

and

$$\boldsymbol{\Pi}^\mu = \frac{\partial \mathscr{L}}{\partial \left(\partial_\mu \boldsymbol{\phi} \right)}. \tag{2.17}$$

After this short introduction to scale invariance we want to investigate which theories are scale invariant and which terms break it. In order to do so we consider the most general renormalizable Lorentz invariant theory in 4 dimensions containing scalars, fermions and gauge bosons and do not specialize it by imposing any gauge symmetry. It can be decomposed into

$$\mathscr{L} = \mathscr{L}_0 + \mathscr{L}_I, \tag{2.18}$$

where

$$\begin{aligned} \mathscr{L}_0 =& \frac{1}{2} \partial_\mu \phi^a \partial^\mu \phi^a - \frac{1}{2} \left(\mu_0^2 \right)^a \phi^a \phi^a + \overline{\psi}^a \left(i \partial^\mu \gamma_\mu - m_0^a \right) \psi^a \\ & - \frac{1}{4} \left(\partial_\mu A_\nu^a - \partial_\nu A_\mu^a \right)^2 + \frac{1}{2} \left(M_0^2 \right)^a A_\mu^a A^{\mu a} \end{aligned} \tag{2.19}$$

and

$$\begin{aligned} \mathscr{L}_I =& \alpha^a \phi^a + \beta^{abc} \phi^a \phi^b \phi^c + \lambda^{abcd} \phi^a \phi^b \phi^c \phi^d \\ & + g^{abc} \overline{\psi}^a \psi^b \phi^c + i h^{abc} \overline{\psi}^a \gamma_5 \psi^b \phi^c + e^{abc} \overline{\psi}^a \gamma^\mu \psi^b A_\mu^c \\ & + 2 f^{abc} \left(\partial^\mu \phi^a \right) \phi^b A_\mu^c + f^{abc} f^{ade} \phi^b \phi^d A_\mu^c A^{\mu e}. \end{aligned} \tag{2.20}$$

Here ϕ denote scalars, ψ fermions and A represent the gauge bosons, where its Latin indices distinguish between different fields respectively and have to be summed over. All the $\alpha, \beta, \lambda, g, h$ denote self-coupling and Yukawa coupling constants, whereas all e and

f are gauge coupling constants. The form of the gauge couplings can be deduced from taking the most general covariant derivative and multiplying it out.
For this Lagrangian we find the scale breaking terms

$$\Delta = \left(\mu_0^2\right)^a \phi^a\phi^a + m_0^a\overline{\psi}^a\psi^a + \left(M_0^2\right)^a A_\mu^a A^{\mu a} - \beta^{abc}\phi^a\phi^b\phi^c - 3\alpha^a\phi^a \,, \tag{2.21}$$

such that $\partial_\mu J^\mu = \Delta$. We immediately see that all terms with dimensionful coupling constants and all mass terms break scale invariance. We can therefore formulate the important rule:

> Any renormalizable theory in 4 dimensions is scale invariant if and only if it contains no dimensionful coupling constants or masses.

2.2.3 Improved Energy-Momentum Tensor

The canonical energy-momentum tensor defined in (2.16) does not have to be symmetric in general. Furthermore, usually it is not found to be renormalizable. Callan et al., however, proved that an energy-momentum tensor can always be defined such that it is symmetric and renormalizable to all orders in perturbation theory [7]. Generally this tensor is of the form [9]

$$\theta^{\mu\nu} = T^{\mu\nu} + \partial_\lambda\Gamma^{\mu\nu\lambda} \,, \tag{2.22}$$

where $\Gamma^{\mu\nu\lambda}$ is antisymmetric under interchange of μ and λ. It also fulfils energy-momentum conservation and yields the same conserved charge as the canonical energy-momentum tensor.[4] This improved energy-momentum tensor plays a crucial role for scale invariance.
Under certain circumstances the scale current J^μ can be written as

$$J^\mu = x_\lambda\theta^{\mu\lambda}, \tag{2.23}$$

where $\theta^{\mu\lambda}$ is the new energy-momentum tensor introduced by Callan et al. This is possible for all theories obeying the condition

$$\boldsymbol{\Pi}^\mu d\boldsymbol{\phi} + \boldsymbol{\Pi}_\lambda\Sigma^{\mu\lambda}\boldsymbol{\phi} = \partial_\lambda\sigma^{\mu\lambda}, \tag{2.24}$$

[4] Energy-momentum conservation is fulfilled because of the antisymmetry of $\Gamma^{\mu\nu\lambda}$ such that $\partial_\mu\partial_\lambda\Gamma^{\mu\nu\lambda} = 0$. That θ yields the same conserved charge can be seen from $\int d^3x\ \theta^{0\nu} = \int d^3x\ T^{0\nu} + \int d^3x\ \partial_\lambda\Gamma^{0\nu\lambda} = \int d^3x\ T^{0\nu} + \int d^3x\ \partial_0\Gamma^{0\nu 0} + \int d^3x\ \partial_i\Gamma^{0\nu i} = \int d^3x\ T^{0\nu}$, where the second term vanishes because of antisymmetry and the third term because the spacial integral over the divergence vanishes.

where $\Sigma^{\mu\lambda}$ is the spin matrix which is defined by the transformation law of the fields under Lorentz transformations via

$$\delta^{\mu\nu}\boldsymbol{\phi} = \left[x^\mu\partial^\nu - x^\nu\partial^\mu + \Sigma^{\mu\nu}\right]\boldsymbol{\phi} \tag{2.25}$$

and $\sigma^{\mu\lambda}$ is an arbitrary tensor. This condition is indeed fulfilled by the most general renormalizable Lagrangian given in eq. (2.18), where

$$\sigma^{\mu\lambda} = \frac{1}{2}g^{\mu\lambda}\boldsymbol{\phi}^a\boldsymbol{\phi}^a. \tag{2.26}$$

It has to be emphasized that this current does not exactly coincide with the one defined in eq. (2.15), but it yields the same spacial integral of its time component as well as the relation $\partial_\mu J^\mu = \Delta$.
From eq. (2.23) we then find

$$\partial_\mu J^\mu = \partial_\mu\left(x_\nu\theta^{\mu\nu}\right) = g_{\mu\nu}\theta^{\mu\nu} + x_\nu\partial_\mu\theta^{\mu\nu} = \theta^\mu_\mu = \Delta\,, \tag{2.27}$$

because $\partial_\mu\theta^{\mu\nu} = \partial_\mu T^{\mu\nu} = 0$, which expresses energy-momentum conservation.
Eq. (2.27) can be summarized in the following way: The scale symmetry breaking part of the Lagrangian is given by the trace of the energy-momentum tensor introduced by Callan et al.

2.2.4 Conformal Transformations

We started studying scale transformations and its properties postponing the introduction of conformal transformations. We are now going to define them and then relate them to scale transformations.
Conformal transformations are space-time transformations as well and are defined as

$$a : x^\mu \to \frac{x^\mu - a^\mu x^2}{1 - 2a_\lambda x^\lambda + a_\lambda a^\lambda x^2}\,, \tag{2.28}$$

where a is a four-vector of real numbers, such that their infinitesimal form is given by[5]

$$\delta^\nu_c x^\mu = -2x^\mu x^\nu + g^{\mu\nu}x^2\,, \tag{2.29}$$

which yields four conserved Noether currents given by

$$J^{\mu\nu} = \left(2x^\mu x^\lambda - g^{\mu\lambda}x^2\right)\theta^\nu_\lambda. \tag{2.30}$$

[5]This can be obtained from a first order Taylor expansion of the right-hand side of eq. (2.28) around $a = 0$.

Applied to fields the infinitesimal conformal transformations given in eq. (2.29) lead to

$$\delta_c^\mu \boldsymbol{\phi} = \left(2x^\mu x^\nu - g^{\mu\nu} x^2\right) \partial_\nu \boldsymbol{\phi} + 2x^\mu d\boldsymbol{\phi} + 2x_\nu \Sigma^{\mu\nu} \boldsymbol{\phi}. \tag{2.31}$$

Infinitesimal conformal and infinitesimal scale transformations are both part of a 15 dimensional Lie algebra which is closed under commutation. Besides one scale and four conformal transformations this Lie algebra further consists of six Lorentz transformations and 4 translations. This means that the commutators of the elements of this algebra are a linear combination of other elements of this algebra. The corresponding group we get when exponentiating the algebra is called the conformal group. The group structure is such that all Lorentz invariant theories holding conformal invariance are automatically scale invariant as well. There is, however, no group theoretical argument for the opposite.

Though, it is possible to find a sufficient condition for a scale invariant theory to be conformally invariant. In the following we will show that, if a theory is scale invariant, the condition given in eq. (2.24) will inevitably lead to four further conserved currents which will be associated with conformal symmetry.

As mentioned before, if eq. (2.24) and scale invariance holds we can find a conserved current J^μ and an energy-momentum tensor such that

$$J^\mu = x_\nu \theta^{\mu\nu}. \tag{2.32}$$

In this case scale invariance implies $\theta^\mu_\mu = 0$. We can then find four other conserved currents given by

$$J^{\mu\nu} = \left(2x^\mu x^\lambda - g^{\mu\lambda} x^2\right) \theta^\nu_\lambda \,, \tag{2.33}$$

because

$$\begin{aligned} \partial_\nu J^{\mu\nu} &= \left(2\delta^\mu_\nu x^\lambda + 2x^\mu \delta^\lambda_\nu - g^{\mu\lambda} 2x_\nu\right) \theta^\nu_\lambda + \left(2x^\mu x^\lambda - g^{\mu\lambda} x^2\right) \partial_\nu \theta^\nu_\lambda \\ &= 2x^\lambda \theta^\mu_\lambda + 2x^\mu \theta^\nu_\nu - 2x_\nu \theta^{\nu\mu} \\ &= 0\,. \end{aligned}$$

These currents are identical to the conformal currents given in eq. (2.30). We can therefore conclude that a theory which is scale invariant and implements condition (2.24) is also conformally invariant. This result is very important for this work. As we are only dealing with renormalizable theories, which generally fulfil this condition, we only have to show that a theory is scale invariant in order to show that it is conformally invariant. We can thus generalize the result from before and say:

Any renormalizable theory in 4 dimensions is conformally invariant if and only if it contains no dimensionful coupling constants or masses.

2.3 The Trace Anomaly

It is now clear under which circumstances a renormalizable theory is conformally invariant from a classical point of view. In a quantum field theory, however, quantum corrections have to be taken into account. We have mentioned that in certain cases classical symmetries can be broken by radiative corrections. These are called anomalies and the Ward-Takahashi identity is not obeyed.
We have worked out that conformal symmetry is strongly coupled to the existence of massive coupling constants and masses. This relation can be understood very intuitively. All massive constants in the theory set a scale and it is therefore comprehensible that the theory is not scale independent. When renormalizing a theory, in any scheme we have to introduce a dimensional quantity as a renormalization scale. Assuming that at this scale all massive terms vanish and the theory is thus classically scale invariant, the quantum induced running of a coupling g causes a shift in the coupling when settling the theory at a different scale. If we shift the renormalization scale by $M \to Me^{\alpha}$ such that $\delta M = \alpha M$, then the shift in the coupling is given by

$$g \to g + \frac{\partial g}{\partial M}\delta M = g + \frac{1}{M}\frac{\partial g}{\partial(\ln M)} \cdot \alpha M = g + \alpha\beta(g) \tag{2.34}$$

where β is the beta function and describes the running of the coupling. This shift in turn generates the change in the Lagrangian

$$\alpha\beta(g)\frac{\partial}{\partial g}\mathscr{L}. \tag{2.35}$$

Thus scale invariance is broken on quantum level such that

$$\partial_\mu J^\mu = \theta^\mu_\mu = \beta(g)\frac{\partial}{\partial g}\mathscr{L}. \tag{2.36}$$

As this anomaly is given by the trace of the energy-momentum tensor it is called trace anomaly.
After all this anomaly is necessary for classically conformally invariant theories to make sense as the world as we know it contains massive particles and scales. It is an interesting idea that all scales in our world are the consequence of radiative breaking of conformal symmetry.

2.4 Bardeen's Argument

In this section we will examine Bardeen's argument for conformally invariant theories to solve the Hierarchy Problem [10]. It provides an alternative to Supersymmetry as the most common solution to the Hierarchy Problem. This feature is the main motivation and the attractiveness of these kind of theories.
To understand Bardeen's argument we first summarize two results concerning conformal symmetry that we obtained before. The first one is that on tree-level conformal invariance for the most general renormalizable theory containing scalars, fermions and gauge bosons is broken by so-called soft terms. These are terms that have mass dimension less than 4 when neglecting coupling constants. The second result is that on loop-level conformal invariance is broken by the beta- function such that

$$\theta^{\mu}_{\mu} = \beta(g_i)\mathcal{O}_i \,, \tag{2.37}$$

where $\mathcal{O}_i$ are dimension 4 operators. Thus the breaking of conformal invariance, namely the conformal anomaly, is accompanied by terms which are not soft and therefore have a different nature. When calculating radiative corrections to the Higgs potential using a cut-off regularization scheme the scale breaking part is given by

$$\theta^{\mu}_{\mu} = \beta(g)\frac{\partial}{\partial g}\mathscr{L} + \text{soft terms involving } \Lambda^2 \,, \tag{2.38}$$

where Λ is the chosen cut-off. The first term on the r.h.s. can be associated with the anomalous breaking of conformal invariance on loop-level. The second term is also a soft term like the terms that classically forbid conformal symmetry. It is therefore argued not to be a radiative effect but to be already there on tree-level. However, we chose the Lagrangian classically scale invariant. Thus there has to be a different explanation for this term. These terms involving the cut-off are indeed the terms responsible for the quadratic divergences of the Higgs mass. Bardeen argues that these terms are a relict of the cut-off regularization scheme which breaks scale invariance by hand. In contrast using dimensional regularization does not yield these quadratic divergences. Thus in order not to spoil scale invariance by the regularization scheme we have to choose counter-terms such that scale invariance is restored in all orders of perturbation theory and the anomalous Ward identity of the theory (eq. (2.36)) is obeyed.
For the solution of the Hierarchy Problem considered from an effective field theoretical point of view see [11]. A consideration involving gravitational couplings can be found in [12].

2.5 Effective Potential and the Coleman-Weinberg Mechanism

Another reason for the attractiveness of the concept of conformal symmetry is the fact that the Standard Model is virtually conformally invariant except for the negative mass term in the Higgs potential

$$V = -\mu^2 \phi^\dagger \phi + \lambda \left(\phi^\dagger \phi\right)^2, \tag{2.39}$$

where ϕ is the Higgs doublet, μ is the massive parameter breaking conformal invariance and λ is the self-coupling constant. Thus we can simply make the Standard Model conformally invariant by forbidding the first term in the potential. It is very intuitive, however, that taking away this only scale of the Standard Model classically avoids electroweak symmetry breaking as the potential does not have a minimum for a non-zero field value. S. Coleman and E. Weinberg, however, showed that for scalar electrodynamics, which does not yield symmetry breaking on tree-level, radiative corrections to the scalar potential indeed shape the effective potential such that it gains a minimum for a non-zero field value and thus leads to spontaneous symmetry breaking [1]. In this section we study the effective potential and its properties as a method to describe radiative corrections in a convenient way. It will be the instrument to investigate if a conformal theory bears radiative symmetry breaking or not.

2.5.1 The Effective Action

There are two ways how the effective action, sometimes called the quantum action or the quantum effective action can be defined. The more demonstrative way of both is by writing it as [6]

$$\begin{aligned}\Gamma(\phi) \equiv &-\frac{1}{2}\int \frac{\mathrm{d}^d k}{(2\pi)^d}\tilde{\phi}(-k)\big(k^2 - m^2 - \Pi(k^2)\big)\tilde{\phi}(k) \\ &+ \sum_{n=3}^{\infty}\int \frac{\mathrm{d}^d k_1}{(2\pi)^d}\cdots\frac{d^d k_n}{(2\pi)^d}(2\pi)^d \delta^d(k_1 + \dots k_n) \\ &\times \mathbf{V}_n(k_1, \dots, k_n)\tilde{\phi}(k_1)\dots\tilde{\phi}(k_n),\end{aligned} \tag{2.40}$$

where ϕ is a scalar field and $\tilde{\phi}(k) = \int \mathrm{d}^d x\, e^{ikx}\phi(x)$.[6] $\Pi(k^2)$ is the loop correction to the scalar propagator consisting of a sum of one-particle irreducible (1PI) diagrams, i.e. diagrams that are still connected if any one line is cut. Similarly the $\mathbf{V}_n$ denote the 1PI n-point vertex functions.

[6] Note that for convenience we consider the one-scalar case only

The term "Effective Action" stems from the fact, that including it in the partition function, the generated tree-level diagrams give the full transition amplitude, implying all loop corrections induced by the classical action. There are now vertices connecting arbitrarily many external lines.
The second more formal way is by defining the effective action as the Legendre transformation [8]

$$\Gamma[\phi] \equiv -\int \mathrm{d}^4x\, \phi(x) J_\phi(x) + W[J_\phi], \tag{2.41}$$

which is a functional of the classical field ϕ. To understand this definition properly one first has to explain the meaning of J_ϕ. To do so let us first make the definition

$$\phi_J(x) \equiv \frac{\langle\Omega|\mathbf{\Phi}(x)|\Omega\rangle_J}{\langle\Omega|\Omega\rangle_J} = -\frac{i}{Z[J]}\frac{\delta}{\delta J(x)} Z[J], \tag{2.42}$$

where $\mathbf{\Phi}$ is the field operator, $|\Omega\rangle$ represents the vacuum state of the interacting theory and the lower J denotes the presence of an external field J. $Z[J]$ is the generating functional of the theory

$$Z[J] \equiv \langle\Omega|\Omega\rangle_J = \int \mathcal{D}\phi \,\exp\left[iS[\phi] + i\int \mathrm{d}^4x\, \phi(x)J(x)\right], \tag{2.43}$$

where $S[\phi]$ is the classical action. $Z[J]$ is the sum of all diagrams in the presence of the external field $J(x)$, including connected as well as disconnected diagrams. It can be seen that this can be written as

$$Z[J] = \sum_{N=0}^{\infty} \frac{1}{N!}(iW[J])^N = \exp\left(iW[J]\right), \tag{2.44}$$

where $iW[J]$ is the sum of all connected diagrams excluding vacuum diagrams.
Having this relation at hand the definition of ϕ_J can now be written as

$$\phi_J(x) = \frac{\delta}{\delta J(x)} W[J]. \tag{2.45}$$

The current $J_\phi(x)$ from eq. (2.41) can now be understood in the following way: For a given field $\phi(x)$, $J_\phi(x)$ is the current in dependence of $\phi(x)$ such that eq. (2.45) is fulfilled. Thus $\Gamma[\phi]$ is a functional of $\phi(x)$.
From the formal definition of the effective action (2.41) we can now derive an important property:

$$\begin{aligned}\frac{\delta\Gamma[\phi]}{\delta\phi(y)} &= -\int \mathrm{d}^4x\, \phi(x)\frac{\delta J_\phi(x)}{\delta\phi(y)} - J_\phi(y)\\ &\quad + \int \mathrm{d}^4x \left[\frac{\delta W[J]}{\delta J(x)}\right]_{J=J_\phi} \frac{\delta J_\phi(x)}{\delta\phi(y)}\end{aligned}$$

Using eq. (2.45) we obtain

$$\frac{\delta\Gamma[\phi]}{\delta\phi(y)} = -J_\phi(y). \tag{2.46}$$

For a vanishing external current this leads to an effective equation of motion for the vacuum field expectation values

$$\frac{\delta\Gamma[\phi]}{\delta\phi(y)} = 0. \tag{2.47}$$

We now have an explanation for the name "Effective Action" in the context of the definition by the Legendre transformation. The solutions for the vacuum expectation values of the fields are given by the stationary points of the effective action. This is exactly the same in classical field theory where solutions for classical fields are given by the stationary points of the classical action $S[\phi]$. The effective action, however, includes all quantum corrections.

Our next task is now to show that both definitions given above are really equivalent. This will be done by showing that the tree-level diagrams generated by the effective action defined by eq. (2.41) give the full scattering amplitude, just like the effective action defined by eq. (2.40) does.

Let us first define the sum of all connected diagrams using the action $g^{-1}\Gamma[\phi]$ instead of the classical action

$$\exp\left(iW_\Gamma[J,g]\right) \equiv \int \mathcal{D}\phi(x) \exp\left[ig^{-1}\left(\Gamma[\phi] + \int \mathrm{d}^4x\, \phi(x)J(x)\right)\right], \tag{2.48}$$

where g is a constant. All propagators P thus contribute a factor g and all vertices V (including external fields) a factor g^{-1} to give the diagram an overall factor of g^{P-V}. For all connected diagrams the number of loops is given by $L = P - V + 1$. So we can write

$$W_\Gamma[J,g] = \sum_{L=0}^{\infty} g^{L-1} W_\Gamma^{(L)}[J], \tag{2.49}$$

where $W_\Gamma^{(L)}[J]$ is the sum of all connected diagrams with L loops and $g = 1$. In the limit $g \to 0$, which corresponds to the transition to purely classical solutions, the path integral is dominated by the point of stationary phase and we obtain the relation

$$\lim_{g\to 0} \exp\left(iW_\Gamma[J,g]\right) \propto \exp\left[ig^{-1}\left(\Gamma[\phi_J] + \int \mathrm{d}^4x\, \phi_J(x)J(x)\right)\right], \tag{2.50}$$

where ϕ_J is given by the classical relation

$$\left.\frac{\delta\Gamma[\phi]}{\delta\phi(x)}\right|_{\phi=\phi_J} = -J(x) \tag{2.51}$$

or equivalently by eq. (2.45).

Taking the logarithm of both sides of eq. (2.50) and comparing terms proportional to

g^{-1} we find

$$W_\Gamma^{(0)}[J] = \Gamma[\phi_J] + \int \mathrm{d}^4x\, \phi_J(x) J(x), \tag{2.52}$$

where $W_\Gamma^{(0)}[J]$ are the 0-loop diagrams, i.e. tree-level diagrams, if we used $\Gamma[\phi]$ as the action. Comparing the right-hand side of this equation with the definition of the effective action (2.41), we get

$$W_\Gamma^{(0)}[J] = W[J]. \tag{2.53}$$

This is exactly what we wanted to show. The sum of all tree-level diagrams taken from the effective action is the same as the sum of all diagrams with arbitrarily many loops taken from the classical action. For more details on the effective action and related subjects see e.g. [13, 14].

2.5.2 The Effective Potential

As we want to use the effective action to understand spontaneous symmetry breaking, which is the case if the vacuum expectation value (vev) of the field is non-zero, we are interested in cases where the vev is constant in space and time as this would otherwise correspond to spontaneous breakdown of momentum conservation. It is then convenient to expand the effective action in powers of momentum

$$\Gamma[\phi] = \int \mathrm{d}^4x \left[-V_{\text{eff}}(\phi) + \frac{1}{2}(\partial_\mu \phi)^2 Z(\phi) + ... \right]. \tag{2.54}$$

V_{eff} is called effective potential. Taking ϕ constant, we get

$$V_{\text{eff}}(\phi_0) = -\frac{\Gamma[\phi_0]}{\mathcal{V}_4}, \tag{2.55}$$

where $\mathcal{V}_4$ is the integral over space and time $\int \mathrm{d}^4x$. In the case of constant fields the solution for the vacuum expectation value in the absence of an external field is thus given by

$$\frac{dV_{\text{eff}}}{d\phi_0} = 0. \tag{2.56}$$

Furthermore we can expand the effective action in terms of ϕ

$$\Gamma[\phi] = \sum_n \frac{1}{n!} \int \mathrm{d}^4x_1 ... \mathrm{d}^4x_n\, \Gamma^{(n)}(x_1, ..., x_n) \phi(x_1) ... \phi(x_n). \tag{2.57}$$

Comparing this expression with eq. (2.40) we see that the coefficients $\Gamma^{(n)}(x_1, ..., x_n)$ are just the 1PI Green's functions in position space. With the definition

$$(2\pi)^4 \delta(k_1 + ... + k_n) \tilde{\Gamma}^{(n)}(k_1, ..., k_n) = \int \mathrm{d}^4x_1 ... \mathrm{d}^4x_n\, \Gamma^{(n)}(x_1, ..., x_n) e^{k_1 x_1 + ... + k_n x_n}, \tag{2.58}$$

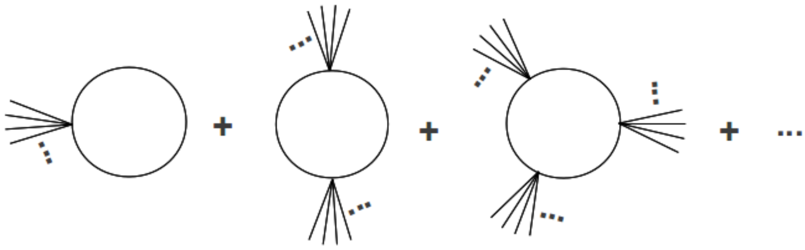


FIGURE 2.1: One-loop contributions to the effective potential

which are the 1PI Green's functions in momentum space, one can see that the effective potential is given by

$$V_{\text{eff}}(\phi_0) = -\sum_{n=0}^{\infty} \frac{1}{n!}(\phi_0)^{(n)}\tilde{\Gamma}^{(n)}(0, ..., 0) \tag{2.59}$$

as the factor $\mathcal{V}_4$ is cancelled by $(2\pi)^4\delta(0)$.

2.5.3 Calculation Example for ϕ^4 Theory

We will now have a closer look at how to actually calculate the effective potential to one loop order. It is clear from eq. (2.59) that considering tree-level Green's functions simply reproduces the classical potential. Our sample computation will be based on a theory of a scalar field with arbitrary potential

$$\mathscr{L} = \frac{1}{2}(\partial_\mu\phi)^2 - V(\phi). \tag{2.60}$$

Note that also the mass term was absorbed by the potential V, which means that we consider propagators without mass in the denominator. All one-loop diagrams with arbitrary number of external lines are given by the diagrams in fig. 2.1.

The external lines will later represent a plug-in for fields at tree-level and thus contribute a factor ϕ_0, while the fields in the loop will be integrated out. So we see that the vertices are given by

$$i\frac{d^2V}{d\phi^2}\bigg|_{\phi=\phi_0} = iV''(\phi_0), \tag{2.61}$$

thus summarizing all different vertices of the theory into one. Every line in the loop has the propagator

$$\frac{i}{k^2 + i\epsilon}, \tag{2.62}$$

where k is the momentum in the loop, which has to be integrated over.
So we can add up all diagrams in fig. 2.1 to get

$$V_{\text{eff}} = V + i \int \frac{\mathrm{d}^4 k}{(2\pi)^4} \sum_{n=1}^{\infty} \frac{1}{2n} \left(\frac{V''(\phi_0)}{k^2 + i\epsilon} \right)^n , \tag{2.63}$$

where the factor $\frac{1}{2n}$ accounts for the internal symmetry of the corresponding diagrams. Evaluating this sum leads to the effective potential

$$V_{\text{eff}} = V + i \int \frac{\mathrm{d}^4 k}{(2\pi)^4} \ln \left(1 + \frac{V''(\phi_0)}{k^2} \right), \tag{2.64}$$

where the integral has been rotated into Euclidean space. This integral is evidently divergent which is no surprise as it is the sum over loop diagrams and we have not yet renormalized the theory. By doing so we first regularize the theory by a momentum cut-off at $k^2 = \Lambda^2$ such that the effective potential converges but depends on Λ. We therefore introduce counter-terms which cancel these dependences. In our case we choose a renormalization scheme called the on-shell scheme which ensures that the parameters in the Lagrangian are given by the physical observables, like the physical mass for example. Furthermore the fields are renormalized such that the residue of the propagator is one, i.e. such that the simple Feynman rules for vertices and propagators can be applied without having to correct for the residue when calculating amplitudes.
Thus including counter-terms and evaluating the effective potential for massless ϕ^4-theory, i.e. for the potential

$$V = \frac{\lambda}{4!}\phi^4 , \tag{2.65}$$

we find

$$V_{\text{eff}} = \frac{\lambda}{4!}\phi_0^4 + \frac{1}{2}B\phi_0^2 + \frac{1}{4!}C\phi_0^4 + \frac{\lambda\Lambda^2}{64\pi^2}\phi_0^2 + \frac{\lambda^2\phi_0^4}{256\pi^2}\left(\ln \frac{\lambda\phi_0^2}{2\Lambda^2} - \frac{1}{2} \right) \tag{2.66}$$

in the approximation $\Lambda^2 \gg \phi^2$, where the second and third term are counter-terms. We now adjust the counter-term parameters B and C according to the on-shell scheme renormalization conditions. In this scheme we want the renormalized mass of the effective potential to vanish. The physical mass is defined as the pole of the propagator and can thus be deduced from eq. (2.59) to be given by

$$\left. \frac{d^2 V_{\text{eff}}}{d\phi_0^2} \right|_{\phi_0 = 0} = 0 , \tag{2.67}$$

which implies

$$B = -\frac{\lambda\Lambda^2}{32\pi^2}. \tag{2.68}$$

Furthermore we renormalize the coupling λ as follows

$$\left.\frac{d^4V_{\text{eff}}}{d\phi_0^4}\right|_{\phi_0=M} = \lambda, \tag{2.69}$$

thus introducing the arbitrary parameter M. We did not renormalize with respect to $\phi_0 = 0$ because the fourth derivative of V_{eff} does not exist at the origin of field space. As a consequence we find

$$C = -\frac{3\lambda^2}{32\pi^2}\left(\ln\frac{\lambda M^2}{2\Lambda^2} + \frac{11}{3}\right), \tag{2.70}$$

which finally yields

$$V_{\text{eff}} = \frac{\lambda}{4!}\phi_0^4 + \frac{\lambda^2\phi_0^4}{256\pi^2}\left(\ln\frac{\phi_0^2}{M^2} - \frac{25}{6}\right). \tag{2.71}$$

Like mentioned above the parameter M is arbitrary and λ can be defined with respect to any other parameter M'. We can use the effective potential in the form of eq. (2.71) to get a relation between the old coupling constant λ and the new one λ' in dependence of M'

$$\lambda' = \left.\frac{d^4V_{\text{eff}}}{d\phi_0^4}\right|_{\phi=M'} = \lambda + \frac{3\lambda^2}{32\pi^2}\ln\frac{M'^2}{M^2}. \tag{2.72}$$

ϕ^4-theory may be the simplest example to demonstrate how to find the effective potential, unfortunately it is not the best example to demonstrate how radiative corrections can induce spontaneous symmetry breaking. When analysing the shape of the effective potential we actually find that the minimum has moved away from zero. A closer look, however, reveals that for field values around the vacuum expectation value, i.e. for values of high interest when talking about symmetry breaking, the perturbation series cannot be trusted as the actual perturbation coefficient is bigger than one. For a more detailed discussion and a way to handle radiative corrections in this case, namely the renormalization group improved effective potential, see appendix A. A more rewarding theory for the demonstration of radiative symmetry breaking is massless scalar electrodynamics which will be discussed in the next section.

2.5.4 The Coleman-Weinberg Mechanism

Coleman and Weinberg showed that for massless scalar electrodynamics which is described by the Lagrangian [1]

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + (D_\mu\phi)^* D^\mu\phi - \frac{\lambda}{6}\left|\phi\right|^4, \tag{2.73}$$

where $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ is the electromagnetic field-strength tensor, $D_\mu = \partial_\mu - ieA_\mu$ is the gauge covariant derivative and A_μ denotes the electromagnetic field, that radiative corrections induce an effective potential which indeed yields spontaneous symmetry breaking and generates massive photons. Furthermore they showed by a renormalization group analysis that the renormalization point can be chosen such that the scalar coupling λ is of order e^4. This ensures that the perturbation coefficient is smaller than one and the perturbation series can actually be trusted.

In order to calculate the effective potential, not only scalar loops but also gauge boson loops have to be taken into account. For a derivation of an arbitrary effective potential and the effective potential of massless scalar electrodynamics see appendix B. It is given by

$$V_{\text{eff}} = \frac{\lambda}{6}|\phi|^4 + \frac{1}{16\pi^2}\left(\frac{5\lambda^2}{18} + 3e^4\right)|\phi|^4\left(\ln\frac{|\phi|^2}{M^2} - \frac{25}{6}\right). \tag{2.74}$$

It is the fact that the renormalization point can be chosen such that $\lambda \propto e^4$ which allows us to neglect the scalar one-loop contribution as it is of order $\lambda^2 \propto e^8$. In the follwoing we thus want to minimize the potential

$$V_{\text{eff}} = \frac{\lambda}{6}|\phi|^4 + \frac{3e^4}{16\pi^2}|\phi|^4\left(\ln\frac{|\phi|^2}{M^2} - \frac{25}{6}\right). \tag{2.75}$$

Doing this yields the relation

$$\lambda(\langle\phi\rangle) = \frac{33}{8\pi^2}e(\langle\phi\rangle)^4, \tag{2.76}$$

where it has to be pointed out that this relation only holds at the renormalization point $M = \langle\phi\rangle$. Plugging this relation into the effective potential gives

$$V_{\text{eff}} = \frac{3e^4}{16\pi^2}|\phi|^4\left(\ln\frac{|\phi|^2}{\langle\phi\rangle^2} - \frac{1}{2}\right). \tag{2.77}$$

At the first glance we see that the dimensionless parameter λ was removed from the effective potential. Naively one could assume that the number of degrees of freedom was reduced. But this is not true. λ was exchanged by the dimensionful parameter $\langle\phi\rangle$. Replacing a dimensionless by a dimensionful parameter is generally called dimensional transmutation and is a consequence of the conformal anomaly.

The mass of the scalar can now be deduced from

$$m_S^2 = \frac{d^2}{d\phi d\phi^*}V_{\text{eff}}\bigg|_{\phi=\langle\phi\rangle} = \frac{3e^4}{4\pi^2}\langle\phi\rangle^2. \tag{2.78}$$

The boson mass is given by

$$m_A^2 = 2e^2\langle\phi\rangle^2, \tag{2.79}$$

which yields

$$\frac{m_S^2}{m_A^2} = \frac{3e^2}{8\pi}. \tag{2.80}$$

2.5.5 CW Mechanism and the Standard Model

As mentioned before, in order to make the Standard Model scale invariant, we only have to forbid the negative mass term in the Higgs potential. After having done that the $SU(2)_L \times U(1)_Y$ symmetry is no more broken on classical level. We can now test if it is broken radiatively. Indeed we find that this is the case under the condition that the Higgs particle has a mass of around 10 GeV and the top mass is smaller than approximately 83 GeV [14]. The latter condition has definitely been ruled out by experiments which reveal the top quark to have a mass of roughly 173 GeV [15]. The former condition is ruled out if the scalar particle found at the LHC at around 125 GeV is indeed the Higgs particle. In any case it is clear that the Standard Model has to be extended by at least one scalar degree of freedom. There have been many attempts to make the SM conformally invariant in a phenomenological acceptable way (see e.g. [16–20]) such that at the same time further problems of physics are addressed. This thesis is supposed to merge the issue of conformal symmetry as a solution to the Hierarchy Problem and the issue of neutrino masses by investigating scenarios that generate neutrino masses in a conformally invariant way and have a particle content such that phenomenological requirements are met.

Chapter 3

The Nature of Neutrinos

For several decades the nature of neutrino masses is an outstanding problem of particle physics. For a very long time neutrinos were assumed to be massless and even today neutrino masses have not been measured directly. However, there is compelling experimental evidence that they must exist. This is implicated by neutrino oscillations, i.e. the observation that neutrinos produced with a certain flavour can transform into a neutrino of different flavour. This is only possible if neutrinos are massive as massless particles do not experience eigentime and can thus not experience intrinsic flavour changes.
In the Standard Model, however, neutrino masses are absent. They cannot be generated like the masses of the charged leptons are as there are no right-handed neutrinos. Furthermore, it is not possible to generate Majorana mass terms for the left-handed neutrinos as there are no lepton number violating couplings. In any case the SM has to be extended to describe the generation of neutrino masses.
This chapter deals with the main properties of neutrinos and how they can be described mathematically in a quantum field theory. The topic of this chapter will be the distinction of Dirac and Majorana neutrinos. It will collect all tools necessary to work with neutrinos and to generate their masses. The explicit formulation of scenarios for neutrino mass generation will be postponed to the next chapters where it will serve as an orientation and comparison for conformally invariant scenarios. Only the most famous mechanism will be discussed as an example.

3.1 Neutrino Oscillations

In this section we will discuss the most important properties of neutrinos, namely the fact that they can oscillate from one into another flavour. It serves as a proof for the existence of small neutrino masses.

Our discussion will be based on the assumption of the existence of sterile neutrinos. This, however, does not limit the generality of the phenomenological considerations. The interaction and mass Lagrangian relevant for neutrino oscillations in the flavour basis is given by

$$-\mathscr{L}_{W+m} = \frac{g}{\sqrt{2}}\overline{l'_{L,\alpha}}\gamma^\mu \nu'_{L,\alpha} W^-_\mu + (m_l)_{\alpha\beta}\overline{l'_{L,\alpha}} l'_{R,\beta} + (m_D)_{\alpha\beta}\overline{\nu'_{L,\alpha}}\nu'_{R,\beta} + h.c.\,, \tag{3.1}$$

where l denotes charged leptons, ν neutrinos of different chirality and W^- the W^--boson. We can then find biunitary transformations to diagonalize m_l and m_D, i.e. such that

$$V_L^\dagger m_l V_R = m_l^d, \quad U_L^\dagger m_D U_R = m_D^d\,. \tag{3.2}$$

The mass eigenstates are thus defined by

$$l'_L = V_L l_L, \quad l'_R = V_R l_R, \quad \nu'_L = U_L \nu_L, \quad \nu'_R = U_R \nu_R\,. \tag{3.3}$$

Consequently the neutrino interaction and mass Lagrangian in the mass basis is given by

$$-\mathscr{L}_{W+m} = \frac{g}{\sqrt{2}}\overline{l_{L,i}}\gamma^\mu (V_L^\dagger U_L)_{ij}\nu_{L,j} W^-_\mu + (m_l^d)_{ii}\overline{l_{L,i}} l_{R,i} + (m_D^d)_{ii}\overline{\nu_{L,i}}\nu_{R,i} + h.c. \tag{3.4}$$

The matrix $U = V_L^\dagger U_L$ is called the lepton mixing matrix. In terms of this matrix we can define a slightly different flavour basis in which unlike above the charged lepton flavour eigenstates and mass eigenstates are identical. This can be achieved by absorbing the charged leptons transformation matrix in a redefinition of the neutrino flavour eigenstates. Thus the new flavour basis eigenstates of the neutrinos can be related to its mass basis via

$$|\nu'_\alpha\rangle = U^*_{\alpha i}|\nu_i\rangle\,. \tag{3.5}$$

Let us now have a flavour eigenstate $|\nu_a\rangle$ at $t = 0$, i.e.

$$|\nu(0)\rangle = |\nu_a\rangle = U^*_{aj}|\nu_j\rangle \tag{3.6}$$

such that we have the time evolution

$$|\nu(t)\rangle = U^*_{aj} e^{-iE_j t}|\nu_j\rangle\,, \tag{3.7}$$

where E_j is the energy of the corresponding mass eigenstate. The amplitude for finding the neutrino at time t in the flavour eigenstate $|\nu_b\rangle$ is then given by

$$A(\nu_a \to \nu_b; t) = \langle \nu_b | \nu(t) \rangle = U^*_{aj} e^{-iE_j t} \langle \nu_b | \nu_j \rangle = U_{bi} U^*_{aj} e^{-iE_j t} \langle \nu_i | \nu_j \rangle = U_{bj} e^{-iE_j t} U^*_{aj} \tag{3.8}$$

and thus the according probability reads

$$P(\nu_a \to \nu_b; t) = |A(\nu_a \to \nu_b; t)|^2 = |U_{bj} e^{-iE_j t} U^*_{aj}|^2 \,. \tag{3.9}$$

To understand the principle properly we first consider the 2 flavour case with the two flavour eigenstates ν_e and ν_μ. The transformation matrix U can then be parametrized in the following way

$$\begin{pmatrix} \cos\theta_0 & \sin\theta_0 \\ -\sin\theta_0 & \cos\theta_0 \end{pmatrix} , \tag{3.10}$$

where θ_0 is the so-called mixing angle, such that we get

$$\begin{aligned} |\nu_e\rangle &= \cos\theta_0 \cdot |\nu_1\rangle + \sin\theta_0 \cdot |\nu_2\rangle \\ |\nu_\mu\rangle &= -\sin\theta_0 \cdot |\nu_1\rangle + \cos\theta_0 \cdot |\nu_2\rangle \end{aligned} \tag{3.11}$$

For relativistic neutrinos of momentum p it holds

$$E_i = \sqrt{p^2 + m_i^2} \approx p + \frac{m_i^2}{2p} \approx p + \frac{m_i^2}{2E} \,. \tag{3.12}$$

In this approximation eq. (3.9) yields

$$P(\nu_e \to \nu_\mu; t) = P(\nu_\mu \to \nu_e; t) = \sin^2(2\theta_0) \sin^2\left(\frac{\Delta m^2}{4E} t\right) , \tag{3.13}$$

where $\Delta m^2 = m_2^2 - m_1^2$. Rewriting it in terms of the travelled distance of the neutrinos we get

$$P(\nu_e \to \nu_\mu; t) = \sin^2 2\theta_0 \sin^2\left(\pi \frac{L}{l_{osc}}\right) , \tag{3.14}$$

where

$$l_{osc} = \frac{4\pi E}{\Delta m^2} \simeq 2.48 km \frac{E(GeV)}{\Delta m^2(eV^2)} \,. \tag{3.15}$$

If we now consider the three flavour case, we have a 3×3 transformation matrix U. In the case of Dirac neutrinos this matrix can be parametrized in dependence of three

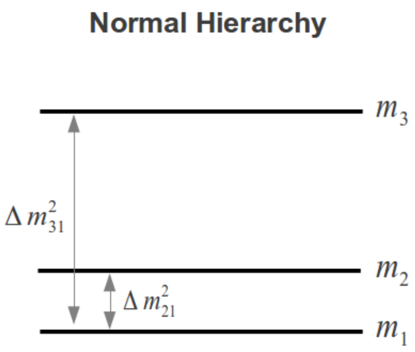


FIGURE 3.1: Normal Hierarchy

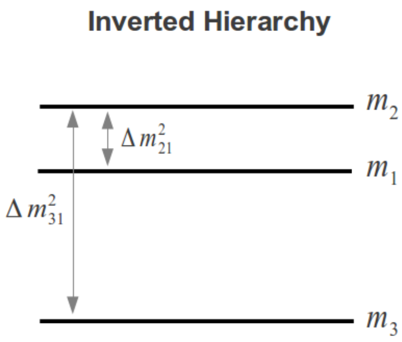


FIGURE 3.2: Inverted Hierarchy

mixing angles θ_{12}, θ_{13}, θ_{23} and one CP-violating phase δ in the following way

$$U = \begin{pmatrix} c_{12}c_{13} & s_{12}c_{13} & s_{13}e^{-i\delta} \\ -s_{12}c_{23} - c_{12}s_{23}s_{13}e^{i\delta} & c_{12}c_{23} - s_{12}s_{23}s_{13}e^{i\delta} & s_{23}c_{13} \\ s_{12}s_{23} - c_{12}c_{23}s_{13}e^{i\delta} & -c_{12}s_{23} - s_{12}c_{23}s_{13}e^{i\delta} & c_{23}c_{13} \end{pmatrix}, \tag{3.16}$$

where s_{ij} and c_{ij} denote $\sin\theta_{ij}$ and $\cos\theta_{ij}$ respectively. Unlike in the two flavour case it is not possible to deduce a simple rule for the transition probability unless we make appropriate assumptions. We will limit ourselves to the consideration of a case that is based on the assumption

$$|\Delta m^2_{21}| \ll |\Delta m^2_{31}| \approx |\Delta m^2_{32}|, \tag{3.17}$$

where $\Delta m^2_{ij} = m^2_i - m^2_j$. As a matter of fact, this assumption has been found to be correct. As it only refers to the absolute values we can have different possible hierarchies. The first one is the Normal Hierarchy (3.1), i.e. $m_1 \ll (\lesssim)\ m_2 \ll m_3$ and the second one is the Inverted Hierarchy (3.1), i.e. $m_3 \ll m_1 \approx m_2$ (see fig. 3.1 and 3.2). If all neutrinos have a mass much larger than the mass differences we speak of the Quasi-Degenerate Hierarchy. In addition to this assumption we further assume small base lines and thus the relation $\frac{\Delta m^2_{21}}{2E} L \ll 1$ holds. In this limit we find for the transition probability between two flavour eigenstates a and b

$$P(\nu_a \to \nu_b; L) = 4|U_{a3}|^2|U_{b3}|^2 \sin^2\left(\frac{\Delta m^2_{31}}{4E} L\right). \tag{3.18}$$

From several oscillation experiments we gain the following mass squared differences (see e.g. [21, 22])

$$\Delta m^2_{21} \approx 7.9 \cdot 10^{-5} eV^2, \quad \Delta m^2_{32} \approx 2.7 \cdot 10^{-3} eV^2. \tag{3.19}$$

For a more detailed introduction to neutrino oscillations and matter effects see [23].

There is one thing remaining to be discussed in more detail. The parametrization of U used above was based on the assumption of Dirac neutrinos and we could reduce the physical parameters to three mixing angles and one CP-violating phase. We will now see how many physical parameters occur in the transformation matrix in different cases. A unitary $n \times n$ matrix depends on $n(n-1)/2$ angles and $n(n+1)/2$ phases. In the Dirac case $2n-1$ phases can be removed by rephasing the left-handed fields, absorbing the effects in the lepton mass terms by corresponding rephasing of the right-handed fields. Hence $(n-1)(n-2)/2$ physical phases remain.
In the Majorana case only n phases can be absorbed such that $n(n-1)/2$ physical phases remain. $(n-1)(n-2)/2$ of those are the Dirac phases. The remaining $n-1$ phases are the so called Majorana phases. For the special case of three flavours we thus have two additional Majorana phases. However, we know nothing about theses phases as we do not even know if neutrinos are of Dirac or Majorana type.

3.2 Dirac and Majorana Neutrinos

3.2.1 Quantum Field Theoretical Description

If we consider electrons for example we observe particles as well as antiparticles of two different helicities. This means that there are 4 degrees of freedom. In a certain reference frame we can describe these particles by a quantum field of the following form

$$\psi(x) = \int \frac{\mathrm{d}^3 p}{\sqrt{(2\pi)^3 2E_p}} \sum_{s=\pm\frac{1}{2}} \left(a_s(\mathbf{p}) u_s(\mathbf{p}) e^{-i\mathbf{px}} + b_s^\dagger(\mathbf{p}) v_s(\mathbf{p}) e^{i\mathbf{px}} \right), \tag{3.20}$$

where its dynamics are classically determined by the Dirac equation and we call these particles Dirac particles. u_s and v_s describe the 4 different helicity states, $a_s(\mathbf{p})$ and $b_s(\mathbf{p})$ are annihilation operators for particles and antiparticles of momentum $\mathbf{p}$ respectively, where their daggered form are the corresponding creation operators.
For neutrinos the situation is different. We only observe particles of left and antiparticles of right helicity. Assuming neutrinos are completely massless there are no questions left and we have two degrees of freedom. The issue of right helicity of neutrinos does not occur as massless particles are moving at the speed of light and there is no boost to turn around helicity. Furthermore, in this case helicity and chirality are identical and thus only neutrinos of left helicity are produced. But as mentioned before, there is strong evidence for neutrino masses and we could assume that neutrinos are just like all other fermions of Dirac type. There is, however, another possibility. E. Majorana suggested

that neutrinos might be there own antiparticles (upto a phase factor) and thus the right-handed neutrino is just the known right-handed antineutrino and the left-handed antineutrino is the left-handed neutrino. In this way we could describe neutrino masses in a Lorentz invariant way with just two degrees of freedom. Thus the corresponding field can be written as

$$\psi(x) = \int \frac{\mathrm{d}^3 p}{\sqrt{(2\pi)^3 2E_p}} \sum_{s=\pm\frac{1}{2}} \left(a_s(\mathbf{p}) u_s(\mathbf{p}) e^{-i\mathbf{px}} + \lambda a_s^\dagger(\mathbf{p}) v_s(\mathbf{p}) e^{i\mathbf{px}} \right), \tag{3.21}$$

where λ is a phase factor. The Majorana condition can be formulated for fields in the following way

$$\psi^c(x) \equiv \gamma_0 C \psi^*(x) = e^{-i\theta}\psi, \tag{3.22}$$

where θ is a phase and C has to be chosen such that the equation is Lorentz invariant and yields the relations

$$\gamma_0 C u_s^*(\mathbf{p}) = v_s(\mathbf{p}) \tag{3.23a}$$

$$\gamma_0 C v_s^*(\mathbf{p}) = u_s(\mathbf{p}). \tag{3.23b}$$

Indeed eq. (3.21) fulfils the Majorana condition with the identification $\lambda = e^{i\theta}$.

3.2.2 Form and Properties of the C Operator

The form of C depends on the representation we choose for the matrices γ_μ, which have to fulfil the anticommutation relation

$$\{\gamma_\mu, \gamma_\nu\} = 2g_{\mu\nu}, \tag{3.24}$$

which is called the Clifford Algebra. Before we can find an explicit representation of C we have to work out the properties it has to meet. First in order to ensure that ψ^c is normalized we have to demand

$$C^\dagger = C^{-1}. \tag{3.25}$$

For eq. (3.22) to be Lorentz invariant it has to be

$$C\sigma_{\mu\nu}^* C^{-1} = -\gamma_0 \sigma_{\mu\nu} \gamma_0, \tag{3.26}$$

where $\sigma_{\mu\nu} = \frac{i}{2}[\sigma_\mu, \sigma_\nu]$ and σ_μ are the Pauli matrices. From this we find the condition

$$C^{-1}\gamma_\mu C = -\gamma_\mu^T. \tag{3.27}$$

From demanding

$$(\psi^c)^c = \psi \tag{3.28}$$

and equations (3.25) and (3.27) we get

$$C^T = -C. \tag{3.29}$$

Together with eqs. (3.23) these conditions uniquely determine the matrix C. Choosing the Dirac representation for the γ matrices

$$\gamma_0 = \begin{pmatrix} 0 & \mathbb{1} \\ \mathbb{1} & 0 \end{pmatrix} \quad , \quad \gamma_i = \begin{pmatrix} 0 & \sigma_i \\ -\sigma_i & 0 \end{pmatrix}, \tag{3.30}$$

we find

$$C = i\gamma_2\gamma_0. \tag{3.31}$$

We see that in this representation the operator C is completely real which yields further useful relations. We thus find

$$C^\dagger = C^T = C^{-1} = -C \tag{3.32}$$

and

$$\overline{\psi^c} = \psi^T C^{-1}, \quad \overline{\psi_L}\psi_R = \overline{(\psi^c)_L}(\psi^c)_R = \overline{(\psi_R)^c}(\psi_L)^c \tag{3.33}$$

There are other representations but we will use the Dirac representation throughout this work.

3.2.3 Neutrinos under Charge Conjugation $\mathcal{C}$

The charge conjugation operation $\mathcal{C}$ is defined by its action on the fields

$$\mathcal{C}\psi(\mathbf{x},t)\mathcal{C}^{-1} = \eta_c^*\gamma_0 C\psi^*(\mathbf{x},t) = \eta_c^*\psi^c(\mathbf{x},t), \tag{3.34}$$

where η_c is a phase factor. This applies for general fermions. For Majorana fields it can be simplified to

$$\mathcal{C}\psi(\mathbf{x},t)\mathcal{C}^{-1} = (\eta_c\lambda)^*\psi(\mathbf{x},t). \tag{3.35}$$

Thus we find

$$\mathcal{C}a_s(\mathbf{p})\mathcal{C}^{-1} = (\eta_c\lambda)^* a_s(\mathbf{p})$$
$$\mathcal{C}a_s^\dagger(\mathbf{p})\mathcal{C}^{-1} = (\eta_c\lambda)^* a_s^\dagger(\mathbf{p}),$$

which yields

$$(\eta_c\lambda)^* = \eta_c\lambda = \pm 1\,,$$

because η_c as well as λ and consequently $\eta_c\lambda$ are mere phase factors. Furthermore, this means that

$$\mathcal{C}|\mathbf{p},s\rangle = \eta_c\lambda|\mathbf{p},s\rangle \equiv \tilde{\eta_c}|\mathbf{p},s\rangle,$$

where $|\mathbf{p},s\rangle = a_s^\dagger(\mathbf{p})|0\rangle$, because $\mathcal{C}|0\rangle = |0\rangle$, i.e. because the vacuum is invariant under charge conjugation.

We conclude that a free Majorana particle is an eigenstate of the charge conjugation operator with eigenvalues $+1$ or -1. The physical Majorana neutrino, however, is not as the interactions violate $\mathcal{C}$.

3.3 Technicalities and Tools

3.3.1 Majorana Basis

A way of expressing the mass terms of a Lagrangian is by writing it as

$$-\mathscr{L}_m = \frac{1}{2}\sum_{a,b}\overline{\psi_{aL}}M_{ab}(\psi_b^c)_R + h.c.\,, \tag{3.36}$$

where M_{ab} is the mass matrix and the ψ_a are 2-component Majorana fields and ψ_a^c are their conjugates. The indices L and R denote left and right chirality respectively. As seen before Majorana particles have 2 degrees of freedom whereas Dirac type particles have twice as many. We will indeed show that a Dirac field can be expressed by 2 Majorana particles and can thus be expressed in the form of eq. (3.36).

Assume we have two Majorana particles ψ and χ and that there are only cross terms of the form

$$\frac{1}{2}m\overline{\psi_L}(\chi^c)_R + \frac{1}{2}m'\overline{\chi_L}(\psi^c)_R + h.c. \tag{3.37}$$

We can then write

$$-\mathscr{L}_m = \frac{1}{2}\left(\overline{\psi_L},\overline{\chi_L}\right)\begin{pmatrix} 0 & m \\ m' & 0 \end{pmatrix}\begin{pmatrix} (\psi^c)_R \\ (\chi^c)_R \end{pmatrix} + h.c. \tag{3.38}$$

If we now set $m' = m$ and make the identification

$$\psi_L \to \Psi_L, \qquad (\chi^c)_R \to \Psi_R,$$
$$(\psi^c)_R \to (\Psi^c)_R, \quad \chi_L \to (\Psi^c)_L,$$

and consider Ψ as a 4-component Dirac field, we get

$$-\mathscr{L}_m = \frac{1}{2}m\left(\overline{\Psi_L}\Psi_R + \overline{(\Psi^c)_L}(\Psi^c)_R\right) + h.c. = m\overline{\Psi_L}\Psi_R + h.c.\,, \tag{3.39}$$

which is the mass term for a Dirac field, where we have used the third identity of eq. (3.33). We learn that in this basis Dirac masses are the off-diagonal entries of the mass matrix.

If we now go backwards and diagonalize the mass matrix from eq. (3.38) we find the mass eigenstates

$$\begin{aligned}\chi_1 &= \frac{1}{\sqrt{2}}\left(\psi_L + (\psi_R)^c\right) + \frac{1}{\sqrt{2}}\left((\psi_L)^c + \psi_R\right)\\ \chi_2 &= \frac{1}{\sqrt{2}}\left(\psi_L - (\psi_R)^c\right) - \frac{1}{\sqrt{2}}\left((\psi_L)^c - \psi_R\right)\,,\end{aligned} \tag{3.40}$$

which are both Majorana neutrinos, where χ_1 has phase $\lambda = +1$ and χ_2 has phase $\lambda = -1$. Thus a Dirac field is the sum of two Majorana fields of opposite phase

$$\psi = \frac{1}{\sqrt{2}}\left(\chi_1 + \chi_2\right). \tag{3.41}$$

3.3.2 Feynman Rules

When we derive the Feynman rules for Dirac fermions rigorously we find the following results. The free field propagator for an incoming particle and an outgoing particle or equivalently for an incoming antiparticle and an outgoing antiparticle is given by

$$\langle 0|\mathcal{T}\left(\psi_A(x)\overline{\psi}_B(y)\right)|0\rangle = \int \frac{\mathrm{d}^4p}{(2\pi)^4}\, e^{ip(x-y)}\left[iS_F(p)\right]_{AB}\,, \tag{3.42}$$

where $S_F(p) = \dfrac{\not{p} + m}{p^2 - m^2 + i\epsilon}$.

For an incoming particle and an outgoing antiparticle and vice versa we find

$$\langle 0|\mathcal{T}\left(\psi_A(x)\psi_B(y)\right)|0\rangle = \langle 0|\mathcal{T}\left(\overline{\psi}_A(x)\overline{\psi}_B(y)\right)|0\rangle = 0. \tag{3.43}$$

For Majorana fermions the case is slightly different. While the propagator in eq. (3.42) is the same for Majorana particles, the propagators in eq. (3.43) are non-zero. That this is true can be seen when considering $\overline{\psi} = \lambda\overline{\psi^c} = \lambda\psi^T C^{-1}$. From this we find $\psi^T = \lambda^*\overline{\psi}C$ and thus

$$\begin{aligned}\langle 0|\mathcal{T}\left(\psi_A(x)\psi_B(y)\right)|0\rangle &= \lambda^* C_{DB}\langle 0|\mathcal{T}\psi_A(x)\overline{\psi}_D(y)|0\rangle\\ &= \lambda^*\int \frac{\mathrm{d}^4p}{(2\pi)^4}\, e^{ip(x-y)}\left[iS_F(p)C\right]_{AB}\end{aligned} \tag{3.44}$$

or in momentum space

$$S_{\psi\psi} = \lambda^* S_F(p)C. \tag{3.45}$$

Furthermore, we find for the free-field propagator of the antiparticle to particle transition

$$\begin{aligned}\langle 0|\mathcal{T}\left(\overline{\psi}_A(x)\overline{\psi}_B(y)\right)|0\rangle &= \lambda(C^{-1})_{DA}\langle 0|\mathcal{T}\psi_D(x)\overline{\psi}_B(y)|0\rangle \\ &= \lambda\int\frac{\mathrm{d}^4p}{(2\pi)^4}\,e^{ip(x-y)}\left[iC^{-1}S_F(p)\right]_{AB}\end{aligned} \tag{3.46}$$

or

$$S_{\bar{\psi}\bar{\psi}} = \lambda C^{-1}S_F(p). \tag{3.47}$$

This finding coincides with the picture of particles and antiparticles being the same. For a more detailed discussion of the mathematical issues of neutrino physics see e.g. [24, 25].

3.4 Neutrino Mass Generation

In the Standard Model neutrinos are massless as there are neither right-handed neutrinos nor lepton-violating terms. The simplest way to induce neutrino masses is thus to introduce right-handed neutrinos. As a consequence the SM symmetries allow us to write down a Majorana mass term for them. Depending on this newly introduced mass scale, which can be chosen arbitrarily as a priori there is no physical mechanism to fix it, different phenomenological scenarios arise where one of those is the famous type I seesaw mechanism [26–30]. This will be discussed in the following.

In this case the Standard Model is extended by 3 right-handed sterile neutrinos. Sterile means that they are singlets under the SM gauge group and thus do not interact via any gauge bosons. The corresponding part of the Lagrangian reads[1]

$$-\mathscr{L}_Y = g_{H,ij}\overline{L_i}\tilde{H}\nu_{R,j} + \frac{1}{2}M_{R,ij}\varphi\overline{\nu_{R,i}}\nu^c_{R,j} + h.c.\,, \tag{3.48}$$

where the L_i are the 3 Lepton doublets, $\tilde{H} = i\sigma_2 H^*$ and H is the Higgs doublet and the $\nu_{R,i}$ are the sterile neutrinos, i.e.

$$L_i = \begin{pmatrix}\nu_{L,i}\\ e_{L,i}\end{pmatrix}, \quad H = \begin{pmatrix}H_+\\ H_0\end{pmatrix}. \tag{3.49}$$

[1]Note that from now on we will use $\nu^c_R = (\nu_R)^c = (\nu^c)_L$

After spontaneous symmetry breaking we find the mass part of the Lagrangian for neutrino masses

$$-\mathscr{L}_m = m_{D,ij}\overline{\nu_{L,i}}\nu_{R,j} + \frac{1}{2}M_{R,ij}\overline{\nu_{R,i}}\nu^c_{R,j} + h.c.\,, \tag{3.50}$$

where

$$m_{D,ij} = g_{H,ij}\cdot\langle H\rangle. \tag{3.51}$$

For reasons of convenience we will from now on consider the one flavour case, while the 3 flavour case will be discussed in detail later.
In the Majorana basis the Lagrangian can then be written as

$$-\mathscr{L}_m = \frac{1}{2}\overline{n_L}\mathcal{M}n^c_L + h.c.\,, \tag{3.52}$$

where $n_L = \begin{pmatrix}\nu_L \\ \nu^c_R\end{pmatrix}$ and the mass matrix $\mathcal{M}$ is given by

$$\mathcal{M} = \begin{pmatrix} 0 & m_D \\ m_D & M_R \end{pmatrix}. \tag{3.53}$$

In order to study the most general case of this matrix we restore the left-handed Majorana entry and thus it reads

$$\mathcal{M} = \begin{pmatrix} M_L & m_D \\ m_D & M_R \end{pmatrix}. \tag{3.54}$$

This matrix is symmetric and can thus be diagonalized by a unitary transformation yielding the mass eigenvalues

$$m_{1,2} = \frac{m_R + m_L}{2} \pm \sqrt{\left(\frac{m_R - m_L}{2}\right)^2 + m_D^2}\,. \tag{3.55}$$

Defining

$$n_L = \begin{pmatrix}\nu_L \\ \nu^c_R\end{pmatrix} = \underbrace{\begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & cos\theta \end{pmatrix}}_{\mathbf{U}} \begin{pmatrix}\chi_{1L} \\ \chi_{2L}\end{pmatrix} \tag{3.56}$$

where U is the orthogonal matrix which diagonalizes $\mathcal{M}$, i.e. $U^T\mathcal{M}U = \mathcal{M}_d$ and $\tan 2\theta = 2m_D/(m_R - m_L)$, we can write

$$\begin{aligned} -\mathscr{L}_m &= \frac{1}{2}\overline{n_L}\mathcal{M}n_L^c + h.c. = \frac{1}{2}\left(m_1\overline{\chi_{1L}}\chi_{1L}^c + m_2\overline{\chi_{2L}}\chi_{2L}^c\right) + h.c. \\ &= \frac{1}{2}(|m_1|\overline{\chi}_1\chi_1 + |m_2|\overline{\chi}_2\chi_2) \end{aligned} \tag{3.57}$$

with the definitions

$$\begin{aligned} \chi_1 &= \chi_{1L} + \eta_1\chi_{1L}^c \\ \chi_2 &= \chi_{2L} + \eta_2\chi_{2L}^c\,, \end{aligned} \tag{3.58}$$

where $\eta_{1,2} = 1$ or -1 for $m_{1,2} > 0$ or < 0 respectively. These are both Majorana particles.

We can now distinguish between several cases in dependence of the entries of the mass matrix $\mathcal{M}$.

case 1: $M_L = M_R = 0$, i.e. the case of pure Dirac neutrinos.

The mass eigenvalues are thus given by

$$m_1 = m_D \quad \text{and} \quad m_2 = -m_D. \tag{3.59}$$

We get

$$\begin{pmatrix} \chi_{1L} \\ \chi_{2L} \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} \nu_L + \nu_R^c \\ \nu_L - \nu_R^c \end{pmatrix} \tag{3.60}$$

and thus

$$\begin{aligned} \chi_1 &= \frac{1}{\sqrt{2}}(\nu_L + \nu_R^c) + \frac{1}{\sqrt{2}}(\nu_L^c + \nu_R) \\ \chi_2 &= \frac{1}{\sqrt{2}}(\nu_L - \nu_R^c) - \frac{1}{\sqrt{2}}(\nu_L^c - \nu_R) \end{aligned} \tag{3.61}$$

and $|m_1| = |m_2| = m_D$.

Like discussed before a Dirac neutrino can be considered as 2 Majorana neutrinos with mass m_D.

case 2: $M_L = 0\,;\ M_R \neq 0$.

case 2.1: $M_R \gg m_D$

$\Rightarrow m_1 \approx -\dfrac{m_D^2}{M_R}$ and $m_2 \approx M_R$.

This is the well-known seesaw mechanism. The largeness of the Majorana mass of the right handed neutrino suppresses the mass of one of the mass eignestates.

case 2.2: $M_R = \epsilon \ll m_D$
$\Rightarrow m_1 \approx m_D + \frac{1}{2}\epsilon$ and $m_2 \approx -m_D + \frac{1}{2}\epsilon$.

We see that if $M_R \neq 0$ the mass spectrum is degenerated and the two Majorana fields cannot be combined to a Dirac neutrino. As in this case, however, ϵ is small, this splitting is small and they still have Dirac properties, e.g. a small contribution to the neutrinoless double β decay. We will call this setup the Pseudo-Dirac scenario.

Chapter 4

Phenomenological Study of Sterile Neutrinos in a Conformally Invariant Theory

In chapter 2 we found that conformal symmetry can be argued to solve the Hierarchy Problem. We then raised the question how theories without explicit mass scale in the Lagrangian can bear massive particles. We found that the quantum induced breaking of conformal invariance, called the conformal or trace anomaly, introduces a scale via dimensional transmutation. This in turn can cause electroweak symmetry breaking which provides a mechanism for the generation of masses in the SM. We encountered the problem that the particle content of the conformal SM does not yield the proper Higgs and top quark mass. The conclusion was that the particle content has to be extended by at least one scalar degree of freedom.
In chapter 3 on the other hand we dealt with the physics of neutrinos and found a phenomenological necessity for an extension of the Standard Model, namely the existence of neutrino masses.
This chapter has two intentions. On the one hand it is supposed to introduce the first conformally invariant way of generating neutrino masses by introducing three right-handed neutrinos and a singlet scalar. On the other hand it has the much more important task to give an overview of the phenomenological implications of the introduction of three sterile neutrinos. Therefore a parameter scan of the involved Yukawa couplings has been performed using different parametrizations and electroweak observables to find viable and excluded regions in this space. The allowed regions will be distinguished by their phenomenological implications. This part is one of the main achievements performed in this work.

4.1 Introduction of the 6×6 Mass Matrix in Conformal Theories

Introducing the 6×6 mass matrix basically means introducing 3 sterile right-handed neutrinos. We studied the effects of the introduction of a sterile neutrino for the one flavour case in the previous chapter. This analysis is now generalized by the extension to the three flavour case and performed within the conformally invariant framework. The situation is complicated by the fact that different flavours can mix among each other. To control these 6×6 matrices different parametrizations can be applied. Two of them will be discussed in this section. Furthermore, the phenomenology of sterile neutrinos will be discussed and a parameter scan for different energy scales of sterile neutrinos will be performed. The results will be summarized.
Before, however, we have to point out the effects of conformal invariance. As conformal symmetry forbids all direct mass terms we are not allowed to write down Majorna mass terms for the right-handed neutrinos

$$M_R \overline{\nu_R} \nu_R^c. \tag{4.1}$$

Thus if we only have three right-handed neutrinos in addition to the Standard Model particle content the mass matrix has the following form

$$\begin{pmatrix} 0 & m_D \\ m_D^T & 0 \end{pmatrix} \tag{4.2}$$

and we have the simple case of Dirac neutrinos. In non-conformal theories this would be possible in general if one is willing to accept Yukawa couplings for neutrinos of the order 10^{-11}. The theory $SM + \nu_R$[1] , however, is not an acceptable conformal theory. We need an additional scalar for the Higgs particle to have the right mass.

4.2 The Meissner-Nicolai Model

K.A. Meissner and H. Nicolai introduced a singlet scalar φ and showed that this is enough to break electroweak symmetry radiatively and that it allows for the correct phenomenology [2]. Their model can be summarized in the following way

Particle content[2]: $L : (2, -1);\ H : (2, 1);\ \nu_R : (1, 0);\ \varphi : (1, 0)$,

[1] This theory extends the Standard Model particle content by three right-handed neutrinos.
[2] Note that only particles and couplings relevant for the generation of neutrino masses are mentioned.

where the first number in brackets denotes the transformation behaviour under the $SU(2)_L$ gauge group, e.g. '2' stands for a $SU(2)_L$ doublet. The second number is the $U(1)_Y$ quantum number. With this particle content we get the

Yukawa Lagrangian: $-\mathscr{L}_Y = g_{H,ij}\overline{L}_i\tilde{H}\nu_{R,j} + \frac{1}{2}g_{\varphi,ii}\varphi\overline{\nu_{R,i}}\nu^c_{R,i} + \text{h.c.}$,

where g_H and g_φ are 3×3 matrices and g_φ was chosen diagonal which can generally be done as the flavour of the three right-handed neutrinos is not coupled to any gauge bosons and can thus be freely redefined. The relevant potential is given by

$$V = \lambda_H(H^\dagger H)^2 + \lambda_\varphi(\varphi^\dagger\varphi)^2 + \lambda_{H\varphi}(\varphi^\dagger\varphi)(H^\dagger H)\,,$$

where the λ are the corresponding non-dimensional coupling constants. If we now assume that the potential yields a non-zero vev for H as well as for φ, i.e.

$$H \longrightarrow \begin{pmatrix} 0 \\ \langle H\rangle \end{pmatrix}, \quad \varphi \longrightarrow \langle\varphi\rangle\,, \tag{4.3}$$

then we obtain from the Yukawa Lagrangian the neutrino mass part of the Lagrangian reading

$$-\mathscr{L}_m = m_{D,ij}\cdot\overline{\nu_{L,i}}\nu_{R,j} + \frac{1}{2}M_{R,ii}\cdot\overline{\nu_{R,i}}\nu^c_{R,i} + h.c.\,, \tag{4.4}$$

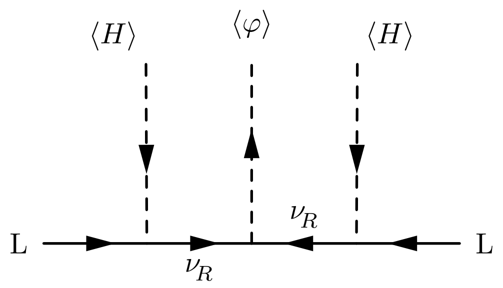
where

$$m_{D,ij} = g_{H,ij}\cdot\langle H\rangle\,, \qquad M_{R,ij} = g_{\varphi,ij}\cdot\langle\varphi\rangle\,. \tag{4.5}$$

Expressing this transition with the Majorana basis matrices we get

$$\begin{pmatrix} 0 & g_H\langle H\rangle \\ g_H^T\langle H\rangle & g_\varphi\langle\varphi\rangle \end{pmatrix} \longrightarrow \begin{pmatrix} 0 & m_D \\ m_D^T & M_R \end{pmatrix}. \tag{4.6}$$

We can also move into a picture where we integrate out the sterile neutrinos and thus get an effective left-handed Majorana neutrino mass. Translated into diagrammatic language this scenario is described by



We see that by introducing a singlet scalar which gains a vev via radiative symmetry breaking, the right-handed Majorana entry of the mass matrix can be restored in a conformally invariant way. In the Meissner-Nicolai model the masses of the active neutrinos depend on the values of g_H, g_φ and $\langle\varphi\rangle$. In dependence of the scales of the Yukawa couplings we can make different approximations and thus choose different parametrizations to describe the diagonalization of the mass matrix.

4.3 Parametrizations

4.3.1 Casas-Ibarra Parametrization

Independent of any scales we can always find a unitary 6×6 matrix $\mathbf{U}$ such that the mass matrix[3] is block-diagonalized

$$\mathbf{U}^T \begin{pmatrix} M_L & m_D \\ m_D^T & M_R \end{pmatrix} \mathbf{U} = \begin{pmatrix} M_l & 0 \\ 0 & M_h \end{pmatrix}, \tag{4.7}$$

where M_l and M_h are 3×3 matrices and the corresponding basis of the matrix on the right-hand side of the equation is given by

$$\begin{pmatrix} \nu_l \\ \nu_h \end{pmatrix} = \mathbf{U}^\dagger \begin{pmatrix} \nu_L \\ \nu_R^c \end{pmatrix}. \tag{4.8}$$

[3]Note that we reintroduce the left-handed Majorana masses in order to consider the most general case for the Casas-Ibarra parametrization. How this can be realized in a conformally invariant theory will be shown later.

The Casas-Ibarra parametrization [31] is based on the assumption that the scale of M_R is much bigger than the scale of m_D, while the one of M_L is much smaller, i.e.

$$\frac{m_D}{M_R} \ll 1 \ll \frac{m_D}{M_L}. \tag{4.9}$$

An adequate parametrization for $\mathbf{U}$ is therefore given by

$$\mathbf{U} = \begin{pmatrix} \sqrt{1 - RR^\dagger} & R \\ -R^\dagger & \sqrt{1 - R^\dagger R} \end{pmatrix}, \tag{4.10}$$

where R is a 3×3 matrix and is treated as a small perturbation. The square root is defined by its Taylor expansion, i.e. $\sqrt{1 - RR^\dagger} \approx 1 - \frac{1}{2}RR^\dagger + \ldots$. To first order R is found to read

$$R \approx \left(M_R^{-1} m_D\right)^\dagger , \tag{4.11}$$

where the light masses are thus given by

$$M_l = M_L - m_D^T M_R^{-1} m_D \tag{4.12}$$

and for the heavy masses we approximately find $M_h = M_R$. This is the seesaw scenario for three flavours as indicated by the assumption (4.9).

The mass matrix M_l, however, is not diagonalized yet. We can find a unitary matrix U such that

$$U^T M_l U = D_l \, , \tag{4.13}$$

where D_l is now diagonal and carries the active neutrino masses. The PMNS matrix is therefore given by

$$U_{\text{PMNS}} = \sqrt{1 - RR^\dagger} \cdot U \approx \left(1 - \frac{1}{2}RR^\dagger\right) U. \tag{4.14}$$

We see that the PMNS matrix lost its unitarity due to additional sterile neutrinos and $RR^\dagger$ can be interpreted as its unitarity violating part. Although this non-unitarity is restricted by oscillation experiments to be very small, it yields phenomenological effects that will be discussed later on. From eq. (4.8) we further find the composition of ν_L,

$$\nu_L = \sqrt{1 - RR^\dagger} \cdot \nu_l + R \cdot \nu_h \tag{4.15}$$

and conclude that the active-sterile mixing is proportional to R.

Observations show that this active-sterile mixing is very small and hence from eq. (4.14) we deduce that U can be approximated by U_{PMNS}. The PMNS matrix will therefore from now on be denoted by U. If we further set $M_L = 0$ we get

$$D_l = -U^T m_D^T M_R^{-1} m_D U = -U^T m_D^T \sqrt{M_R^{-1}} \sqrt{M_R^{-1}} m_D U \,, \tag{4.16}$$

where again it has to be noted that M_R was chosen diagonal such that $\sqrt{M_R^{-1}}$ is naturally defined as the diagonal matrix with the inverse square roots of the M_R entries. We further conclude

$$\mathbb{1} = \left[i\sqrt{M_R^{-1}} m_D U \sqrt{D_l^{-1}} \right]^T \underbrace{\left[i\sqrt{M_R^{-1}} m_D U \sqrt{D_l^{-1}} \right]}_{\mathscr{O}} , \tag{4.17}$$

where this equation is solved for any orthogonal matrix $\mathscr{O}$. In dependence of this matrix, the Dirac matrix can be expressed as

$$m_D = i\sqrt{M_R} \mathscr{O} \sqrt{D_l} U^\dagger . \tag{4.18}$$

The advantage of this parametrization is that the theoretical parameters are split up into the phenomenological, i.e. the light neutrino masses in D_l and the mixing angles in the PMNS matrix U, left over theoretical, i.e. the right-handed masses in M_R, and 6 parameters contained in $\mathscr{O}$. These latter 6 parameters are said to have no phenomenological relevance. We can, however, find the relation

$$R = \left(M_R^{-1} m_D \right)^\dagger = -iU \sqrt{D_l} \mathscr{O}^\dagger \sqrt{M_R^{-1}} \,, \tag{4.19}$$

which establishes a connection between $\mathscr{O}$ and the active-sterile mixing R.
Within this parametrization it is easy to scan the space of theoretical parameters such that only the subspace that yields the right masses and mixing angles is considered.

4.3.2 Parametrization for the Pseudo-Dirac Case

In the limit of $M_L, M_R \ll m_D$, where this inequality refers to the scales of the matrices, we generate Pseudo-Dirac neutrinos. The phenomenology of these kind of neutrinos will be discussed in this section. In this case we cannot use the Casas-Ibarra parametrization and we have to proceed differently. The parametrization presented here is based on ref. [32]
First of all we do not diagonalize the mass matrix itself but rather

$$\mathcal{M}^\dagger \mathcal{M} \approx \begin{pmatrix} m_D^\dagger m_D & m_D^\dagger M_R \\ M_R m_D & m_D^* m_D^T \end{pmatrix} , \tag{4.20}$$

where we already neglected higher order terms like e.g. M_R^2. Note that $\mathcal{M}^\dagger\mathcal{M}$ is hermitian by construction. We go on by first diagonalizing the Dirac matrix m_d which can be performed as usual by a biunitary transformation such that

$$U_R^\dagger m_D U_L = \hat{m}_D\,, \tag{4.21}$$

where $\hat{m}_D$ is diagonal. Thus the basis of the diagonal matrix is given by

$$\hat{\nu}_L = U_L^\dagger \nu_L\,. \tag{4.22}$$

U_L can be considered as the PMNS matrix and will be called U in the following. We now define the unitary matrix

$$V = \begin{pmatrix} U & 0 \\ 0 & U_R^* \end{pmatrix}, \tag{4.23}$$

which, if the neutrinos were purely Dirac fermions, would completely diagonalize the matrix $\mathcal{M}^\dagger\mathcal{M}$. In our case, however, we find

$$V^\dagger(\mathcal{M}^\dagger\mathcal{M})V = \begin{pmatrix} \hat{m}_D^2 & \hat{m}_D U_R^\dagger M_R U_R^* \\ U_R^T M_R U_R \hat{m}_D & \hat{m}_D^2 \end{pmatrix}. \tag{4.24}$$

Although the non-diagonal entries are very small compared to the diagonal block-entries, we cannot simply neglect them as these are responsible for breaking the degeneration of the mass eigenvalues. To find the mass eigenvalues we only have to consider the elements of the matrix in eq. (4.24) which connect the initially degenerated couples of mass eigenvalues. These are just the diagonal entries of the off-diagonal blocks. We therefore effectively find three 2×2 matrices of the form

$$\begin{pmatrix} m_i^2 & m_i\epsilon_i^* \\ m_i\epsilon_i & m_i^2 \end{pmatrix}, \tag{4.25}$$

where the m_i are the diagonal entries of the matrix $\hat{m}_D$ and $\epsilon_i \equiv (U_R^T M_R U_R)_{ii}$. Note that $|\epsilon_i| \ll m_i$. These 3 matrices yield 3 pairs of mass eigenstates given by

$$\nu_{S,i} \equiv \frac{1}{\sqrt{2}}\left(\nu_{L,i} + e^{i\phi_i}\nu_{L,i}^c\right), \quad \nu_{A,i} \equiv \frac{1}{\sqrt{2}i}\left(\nu_{L,i} - e^{i\phi_i}\nu_{L,i}^c\right), \tag{4.26}$$

where $e^{i\phi_i} = \frac{\epsilon_i}{|\epsilon_i|}$. The corresponding mass eigenvalues read

$$m_{S,i}^2 = m_i^2 + m_i|\epsilon_i|, \quad m_{A,i}^2 = m_i^2 - m_i|\epsilon_i|. \tag{4.27}$$

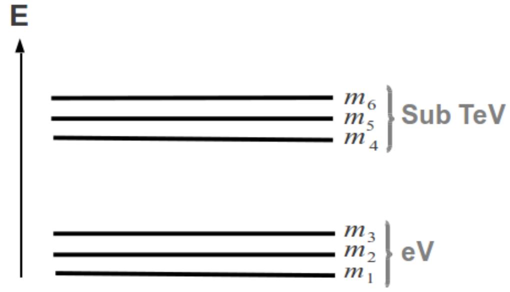


FIGURE 4.1: Seesaw mass spectrum in conformal framework.

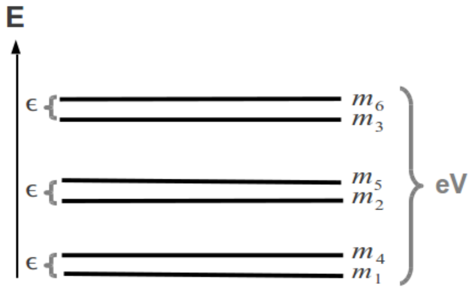


FIGURE 4.2: Pseudo-Dirac mass spectrum.

This is the main characteristic of Pseudo-Dirac neutrinos. The masses of a pair are slightly different such that they cannot be combined to a Dirac neutrino. The full unitary matrix $\hat{V}$ which fulfils the relation

$$\hat{V}^\dagger \left(\mathcal{M}^\dagger \mathcal{M}\right) \hat{V} = M_d^2 \,, \tag{4.28}$$

where M_d^2 is the fully diagonalized 6×6 mass matrix is thus given by

$$\hat{V} = \begin{pmatrix} U & 0 \\ 0 & U_R^* \end{pmatrix} \cdot \frac{1}{\sqrt{2}} \begin{pmatrix} \mathbb{1} & i \cdot \mathbb{1} \\ D & -iD \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} U & iU \\ U_R^* D & -iU_R^* D \end{pmatrix}, \tag{4.29}$$

where $D = \mathrm{diag}(e^{-i\phi_1}, e^{-i\phi_2}, e^{-i\phi_3})$. From eq. (4.28) we then find

$$\mathcal{M}^\dagger \mathcal{M} = \hat{V} M_D^2 \hat{V}^\dagger. \tag{4.30}$$

Evaluating the right-hand side and comparing the upper left blocks gives the approximation

$$m_D^2 = U \hat{m}_D^2 U^\dagger. \tag{4.31}$$

This result can immediately be derived from eq. (4.21). We see that in the Pseudo-Dirac case, the scale of the mass eigenstates comprised of mainly active neutrino states does not depend on the right-handed masses M_R. The Dirac masses rather determine the central mass of the mass eigenvalue pairs while M_R decides how big the splitting between the members of this pair is. For a distinction of the seesaw and the Pseudo-Dirac mass spectrum see fig. (4.1) and (4.2).

4.4 Phenomenology of Heavy Sterile Neutrinos

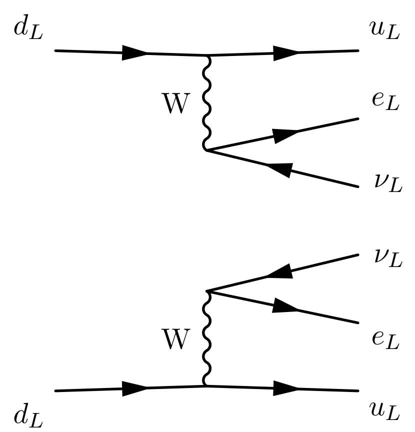
Introducing new particles to the Standard Model is always accompanied by a change of the expected phenomenology. Consequently the predictions of the extended theory have to be compared with observations. In this section we describe the phenomenological consequences of the introduction of sterile right-handed neutrinos in a non-conformal way or equivalently in a conformal way like done within this chapter. These phenomenological findings are confronted with experimental results which in our parameter scan will serve as boundaries for viable regions in parameter space. Basically there are two main properties of theories including right-handed neutrinos. One is based on the Majorana character of right-handed neutrinos and one is the non-unitarity of the resulting effective Lagrangian for active neutrinos. Further phenomenological implications of sterile neutrinos can be found in [33–35].

Neutrinoless Double Beta Decay

In contrast to the neutrinoless double beta decay ($2\beta 0\nu$) there is the double beta decay $2\beta 2\nu$ which is allowed in the Standard Model. There two neutrons or two protons in the nucleus decay into two protons or neutrons, two electrons or positrons and two neutrinos or antineutrinos respectively, resulting in the atomic transition

$$A(Z,N) \longrightarrow A(Z \pm 2, N \mp 2) + 2e^{\mp} + 2\overline{\nu}_e(2\nu_e)\,, \tag{4.32}$$

where A denotes the nucleus, Z is the proton number and N is the number of neutrons. The double neutron decay is described by the diagram

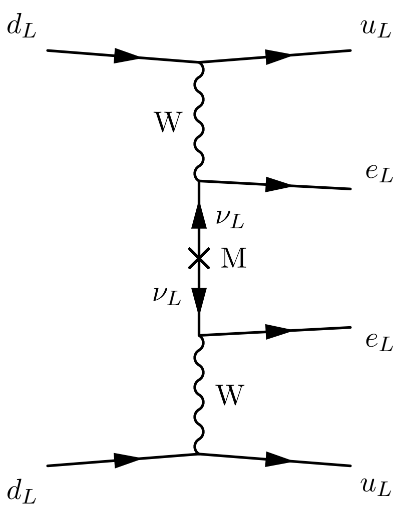


This process, however, is found to have a lifetime of $T \gtrsim 10^{19}$ years.
The neutrinoless double beta decay on the other hand is not possible within the SM

as it violates lepton number conservation. Roughly speaking, in this process a neutrino produced in one decay can be absorbed as an antineutrino in the other decay. The decay is defined by

$$A(Z,N) \longrightarrow A(Z \pm 2, N \mp 2) + 2e^{\mp} \tag{4.33}$$

and is described by the diagram



As the nucleus is very heavy compared to the electrons, its recoil can be neglected and the total energy is absorbed by the electrons. Thus the energy profile of the electrons is a discrete line which is searched for in experiments like GERDA [36].

The amplitude of this process is found to be proportional to the effective mass of the electron-neutrino which is defined by

$$\langle m_{ee}\rangle \equiv \textstyle\sum_i \mathbf{U}_{ei}^2 m_i\,, \tag{4.34}$$

where m_i denotes the mass eigenvalues and $\mathbf{U}_{ei}$ are the corresponding entries of the full 6×6 matrix which transforms between flavour and mass eigenstates.

In general the mass parameters can be positive or negative. Dirac pairs consist of mass degenerate Majorana particles with opposite phases such that their contributions to the effective mass cancel exactly and the amplitude is zero as expected. Similarly, in the Pseudo-Dirac case the terms almost cancel and the neutrinoless double beta decay is very much suppressed. The upper bound of the effective mass has been placed at around 0.2 eV [36].

In the special case of heavy sterile neutrinos we find the approximation [37]

$$\langle m_{ee}\rangle \approx \textstyle\sum_{i=1}^3 \mathbf{U}_{ei}^2 m_i - \sum_{i=4}^6 F(A,m_i)\mathbf{U}_{ei}^2 m_i\,. \tag{4.35}$$

For TeV neutrinos we can use $F(A,m_i) \approx (m_a/m_i)^2 f(A)$, where $m_a \approx 0.9$ GeV and $f(A)$ depends on the isotope [38, 39].

Non-Unitarity of the PMNS Matrix

Due to the introduction of sterile neutrinos the PMNS matrix becomes non-unitary as it has been explained before. The relevant part of the effective interaction Lagrangian in the mass basis reads

$$\begin{aligned}\mathscr{L}_{\text{int}} = &-\frac{e}{2c_w s_w} Z_\mu \sum_{i,j=1}^{3} \sum_{\alpha=e,\mu,\tau} \overline{\nu}_{L,i} \mathbf{U}^\dagger_{i\alpha} \gamma^\mu \mathbf{U}_{\alpha j} \nu_{L,j} \\ &- \frac{e}{\sqrt{2} s_w} W_\mu \sum_{i=1}^{3} \sum_{\alpha=e,\mu,\tau} \overline{\nu}_{L,i} \mathbf{U}^\dagger_{i\alpha} \gamma^\mu l_{L,\alpha} + \text{h.c.}\,. \end{aligned} \tag{4.36}$$

where U is the non-unitary PMNS matrix, l_L are the charged leptons, c_w and s_w stand for $\cos\theta_w$ and $\sin\theta_w$ respectively, θ_w is the Weinberg angle and Z and W denote Z- and W-Boson respectively. The heavy right-handed neutrinos have been integrated out. From this Lagrangian we can calculate the decay widths for the W-bosons into charged leptons and neutrinos

$$\Gamma(W \to l_\alpha \nu_\alpha) = \frac{G_F M_W^3}{6\sqrt{2}\pi} (UU^\dagger)_{\alpha\alpha} \approx \frac{G_F M_W^3}{6\sqrt{2}\pi} (1 - RR^\dagger)_{\alpha\alpha}\,, \tag{4.37}$$

where $G_F = \frac{\sqrt{2} g^2}{8 M_W^2}$ and M_W is the W-mass. The measurements of these decays give information about the diagonal entries of the non-unitarity of the PMNS matrix.
Furthermore, there is missing energy in the decay of the Z-boson. This energy is carried by neutrinos being produced in a Z decay. These are called invisible Z decays and the amplitude is given by

$$\Gamma(Z \to \text{invisible}) = \sum_{i,j} \Gamma(Z \to \overline{\nu}_i \nu_j) = \frac{G_F M_Z^3}{12\sqrt{2}\pi} (1 + \rho_t) \sum_{i,j} |(U^\dagger U)_{ij}|^2\,, \tag{4.38}$$

where the factor $\rho_t \approx 0.008$ accounts for radiative corrections.
Another implication of heavy sterile neutrinos are lepton universality violations. To describe what this means we define a measure for the non-unitarity of the PMNS matrix by

$$\epsilon_\alpha \equiv \sum_{i=1}^{3} |R_{\alpha i}|^2. \tag{4.39}$$

The values for $\alpha = e, \mu, \tau$ theoretically do not have to be the same. If they are not we say that lepton universality is violated. There are several universality tests, e.g. the ratio

$$\frac{\Gamma(\tau \to \nu_\tau \mu \overline{\nu}_\mu)}{\Gamma(\tau \to \nu_\tau e \overline{\nu}_e)}$$

compares two tau-decays into lepton pairs of different flavour. From this ratio we can deduce

$$\frac{(UU^\dagger)_{\mu\mu}}{(UU^\dagger)_{ee}} = \frac{\epsilon_\mu}{\epsilon_e} = 0.9999 \pm 0.0020\,.$$

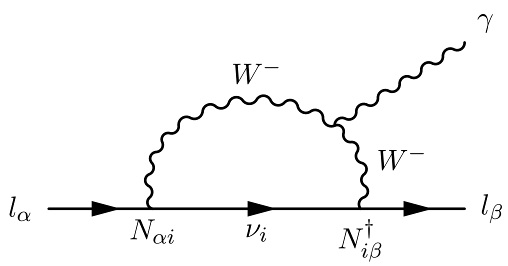
From a collection of these kind of ratios the following universality bounds can be found [40]

$$\epsilon_e - \epsilon_\mu = 0.0022 \pm 0.0025\,, \tag{4.40a}$$

$$\epsilon_\mu - \epsilon_\tau = 0.0017 \pm 0.0038\,, \tag{4.40b}$$

$$\epsilon_e - \epsilon_\tau = 0.0039 \pm 0.0040\,. \tag{4.40c}$$

The last implication of sterile neutrinos we want to consider are rare charged lepton flavour violating decays $l_\alpha \to l_\beta \gamma$:



The branching ratio between this decay and the lepton number conserving decay $l_\alpha \to \nu_\alpha l_\beta \overline{\nu}_\beta$ is given by

$$\mathrm{BR}(l_\alpha \to l_\beta\gamma) = \frac{\Gamma(l_\alpha \to l_\beta\gamma)}{\Gamma(l_\alpha \to \nu_\alpha l_\beta \overline{\nu}_\beta)} = \frac{3\alpha}{32\pi}\frac{|(UU^\dagger)_{\alpha\beta}|^2}{(UU^\dagger)_{\alpha\alpha}(UU^\dagger)_{\beta\beta}}. \tag{4.41}$$

This decay gives delivers information about the non-diagonal entries of the non-unitarity. For the parameter scan, however, we will use the full formula, which takes into account the propagation of the heavy states in the loop. It then reads especially for the transition $\mu \to e\gamma$ [41]

$$\mathrm{BR}(\mu \to e\gamma) = \frac{3\alpha}{32\pi}|\delta_\nu|^2\,, \tag{4.42}$$

where

$$\delta_\nu = 2\sum_i \mathbf{U}^*_{ei}\mathbf{U}_{\mu i}\, g\left(m_i^2/M_W^2\right) \approx 2\textstyle\sum_{i=4}^{6}\mathbf{U}^*_{ei}\mathbf{U}_{\mu i}\left[g\left(m_i^2/M_W^2\right) - 5/3\right]\,, \tag{4.43}$$

where $\mathbf{U}$ is now the full unitary transformation matrix and $g(x)$ is the so called loop function

$$g(x) = \int_0^1 \frac{(1-\alpha)d\alpha}{(1-\alpha)+\alpha x}[2(1-\alpha)(2-\alpha)+\alpha(1+\alpha)x]\,. \tag{4.44}$$

In the last step we used that for the light eigenstates we find $\frac{m_i^2}{M_W^2} \ll 1$ and hence we used $g(m_i^2/M_W^2) \approx g(0) = 5/3$. Beyond that we used that the unitarity of $\mathbf{U}$ yields $\sum_{i=1}^{6} \mathbf{U}_{ei}^* \mathbf{U}_{\mu i} = 0$.

4.5 Parameter Scan for Sub TeV Neutrinos

The attractiveness of the Meissner-Nicolai model lies in the fact that we do not have to introduce a new mass scale. We rather generate the right-handed Majorana masses via electroweak symmetry breaking which can be considered more natural. In this way we set an upper limit to the mass of the heavy sterile neutrinos which will be of electroweak scale. This means that we consider maximally TeV neutrinos. This is because radiative symmetry breaking naturally generates vevs of similar scales such that we assume the vev of the singlet scalar to be around 1 TeV. As we further assume the theory to be perturbative the coupling constants may not be much bigger than 1. Thus the scale of the heavy neutrinos is around 1 TeV maximally. The only relevant parameters for neutrino masses are thus given by the vacuum expectation values of the Higgs field and the newly introduced singlet scalar and the Yukawa couplings of Dirac and Majorana type.

If we considered only Pseudo-Dirac neutrinos, current upper limits for neutrino masses would require the corresponding Yukawa couplings to be below 10^{-11}. It is a common opinion to consider this fact as unnatural. In the framework of the Meissner-Nicolai model we therefore ask the question which regions of parameter space of Dirac and Majorana Yukawa couplings are phenomenologically allowed and how these couplings can be arranged such that they are as close as possible to 1 and approach each other maximally. We will see how this approach happens at the cost of non-unitarity of the PMNS matrix. Furthermore, we will show how experimental bounds set limits to the viable regions in parameter space. The experimental results taken into account are the bounds on the branching ratio of the $\mu \to e\gamma$ decay, the effective electron neutrino mass and lepton universality bounds. Assumed neutrino masses for different hierarchy models and mixing angles have been implemented within the parametrization.

We saw that for heavy sterile neutrinos the Casas-Ibarra parametrization is advantageous as it separates the phenomenological degrees of freedom, i.e. neutrino masses and mixing angles, obtained from oscillation experiments. In the Meissner-Nicolai scenario eq. (4.18) can be written as

$$Y_D = i\frac{\sqrt{\langle\varphi\rangle}}{\langle H\rangle}\sqrt{Y_M}\mathscr{O}\sqrt{D_l}U^\dagger\,, \tag{4.45}$$

which now provides a relation between the matrix of Dirac Yukawa couplings Y_D and the matrix of Majorana Yukawa couplings Y_M. This relation, however, is not unique because of the 6 free parameters in $\mathscr{O}$. These can be parametrized by three complex angles each being associated to a rotation matrix. The product of these three rotation matrices then forms $\mathscr{O}$. With eq. (4.19) we obtain

$$R = \frac{-i}{\sqrt{\langle\varphi\rangle}} U \sqrt{D_l} \mathscr{O}^\dagger \sqrt{Y_M^{-1}} \,. \tag{4.46}$$

Based on this our C++ program produced 100,000 random values for the six free parameters of the orthogonal matrix and for the right-handed Majorana masses in certain intervals and calculated the corresponding Dirac Yukawa matrix Y_D and active-sterile mixing R. From the active-sterile mixing it then determined the measure for the non-unitarity given by the quadratic sum of all entries of R. The real parameters of $\mathscr{O}$ were randomly chosen between 0 and 2π, whereas the complex parameters lie between -16 and 16. The values are distributed linearly over the intervals. For the right-handed masses we chose a logarithmic distribution between $10^{-2.5}$ and $10^{3.5}$ GeV, where they were chosen not to differ by more than one order of magnitude among each other. We arbitrarily chose the vev of the singlet scalar φ to be 1 TeV which translates for the possible Majorana Yukawa couplings to lie between $10^{-5.5}$ and $10^{0.5}$. The particular Yukawa couplings haven then been averaged over to yield one average number for the Dirac couplings and one for the Majorana couplings. This averaging has been done as we are most interested in how the scale of Majorana and Dirac coupling constants are related to each other. We are not interested in detailed information about the biggest or the smallest value of the couplings. Furthermore the relation between right-handed masses has been chosen such that there will be no large gaps between smallest and biggest values. These two numbers then have been plotted in a 2D map together with their respective value of non-unitarity which is expressed by a color scheme. The entries of U which can, to first order, be identified with the PMNS matrix, are determined by oscillation experiments and their best fit values and 1σ ranges are given by [42]

$$\sin^2\theta_{12} = 0.30 \pm 0.013 \tag{4.47a}$$
$$\sin^2\theta_{23} = 0.41^{+0.037}_{-0.025} \tag{4.47b}$$
$$\sin^2\theta_{13} = 0.023 \pm 0.0023 \tag{4.47c}$$
$$\delta_{CP} = 300^{+66}_{-138} \,, \tag{4.47d}$$

where the θ_{ij} denote the mixing angles and δ_{CP} denotes the CP-phase. The two Majorana phases have been set to zero as these are unknown. The neutrino masses are not known exactly but only the mass squared differences. Therefore we examine three different mass hierarchy models namely the Normal Hierarchy with the mass values

$$m_1 \approx 0\text{eV}; \quad m_2 = 8.660 \cdot 10^{-3}\text{eV}; \quad m_3 = 4.97 \cdot 10^{-3}\text{eV} \,, \tag{4.48}$$

the Inverted Hierarchy, where

$$m_1 = 4.85 \cdot 10^{-2}\text{eV}; \qquad m_2 = 4.93 \cdot 10^{-2}\text{eV}; \qquad m_3 \approx 0\text{eV}\,, \tag{4.49}$$

and the Quasi-Degenerate Hierarchy given by

$$m_1 \approx 0.1\text{eV}; \qquad m_2 \approx 0.1\text{eV}; \qquad m_3 \approx 0.1\text{eV}\,. \tag{4.50}$$

Until now our map would show 100,000 coloured points all yielding the right active neutrino masses, mixing angles and CP-phase. These points, however, are reduced by the requirements shown in the previous section. We will only accept points which yield the right effective electron neutrino mass, lepton universality bounds and $\mu \to e\gamma$ branching ratio. All other points are dismissed.

In figures 4.3, 4.4 and 4.5 we show the results of the parameter scan for Normal, Inverted and Quasi -Degenerate Hierarchy. First of all we observe that there is no fundamental difference between the three cases. Scales, the shape of the allowed region and the boundaries are all qualitatively identical.

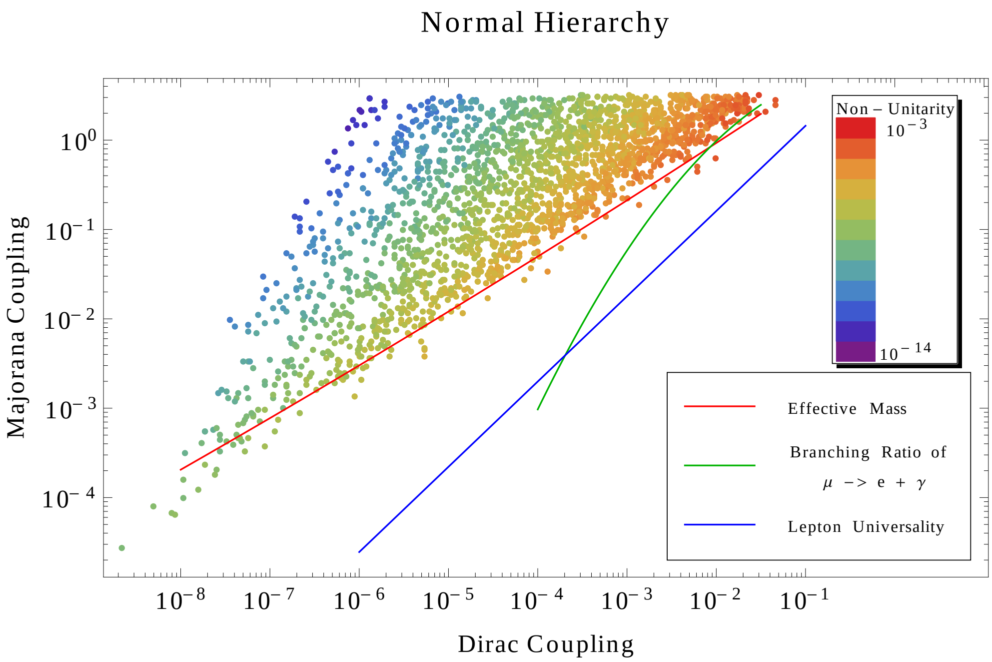


Figure 4.3: Phenomenological viable points in the space of Dirac and Majorana Yukawa couplings for the Normal Hierarchy. The colour of the points represents the corresponding degree of non-unitarity of the PMNS matrix. The lines describe the borders to forbidden regions set by different phenomenological bounds.

Basically it is evident that there is no unique relation between Majorana and Dirac couplings. The bigger e.g. the Majorana coupling the broader is the interval of possible Dirac couplings. This in turn means that there is a minimum in the Majorana and Dirac

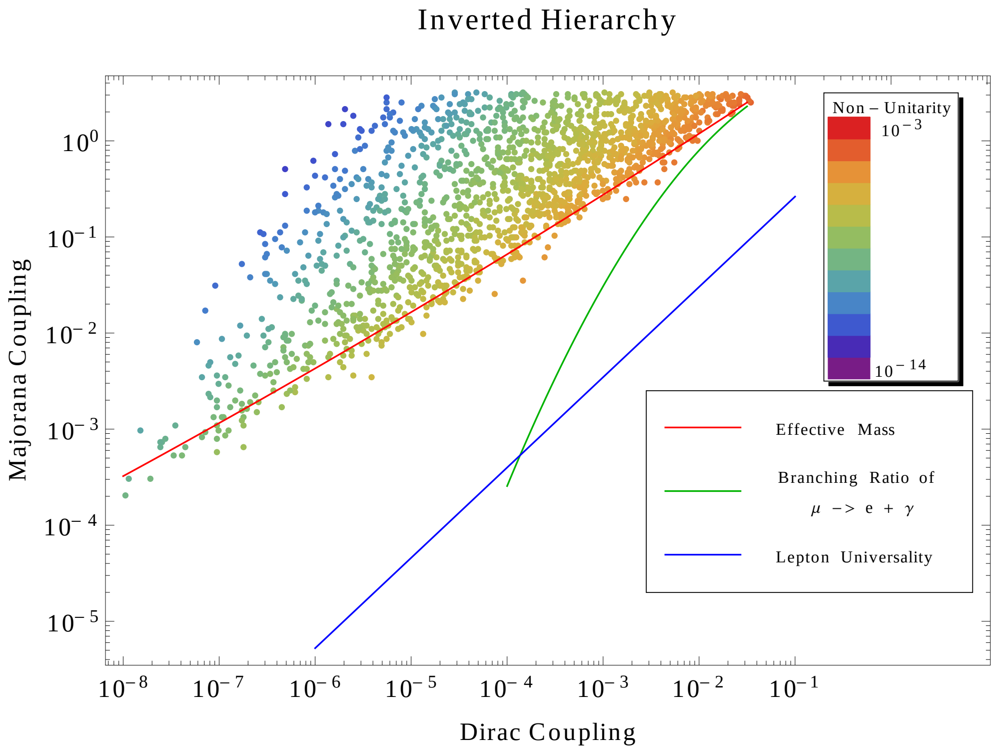


FIGURE 4.4: Phenomenological viable points in the space of Dirac and Majorana Yukawa couplings for the Inverted Hierarchy. The colour of the points represents the corresponding degree of non-unitarity of the PMNS matrix. The lines describe the borders to forbidden regions set by different phenomenological bounds.

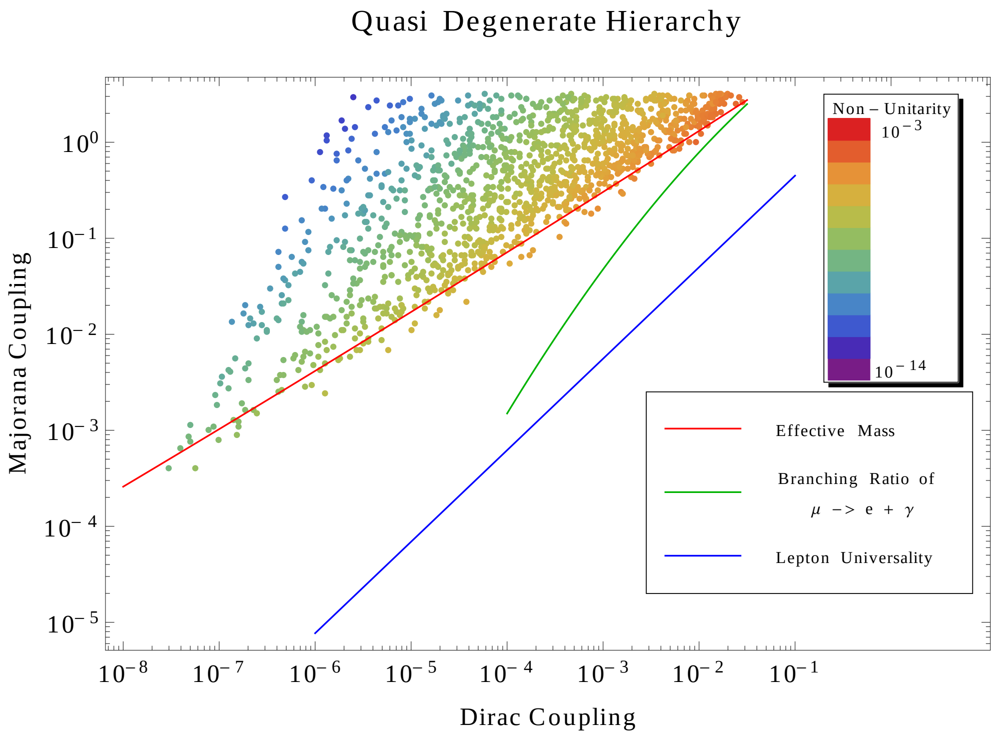


FIGURE 4.5: Phenomenological viable points in the space of Dirac and Majorana Yukawa couplings for the Quasi Degenerate Hierarchy. The colour of the points represents the corresponding degree of non-unitarity of the PMNS matrix. The lines describe the borders to forbidden regions set by different phenomenological bounds.

couplings. We can say that the Dirac coupling will not be smaller than 10^{-9} and the Majorana coupling not smaller than 10^{-5}. The broadening of the allowed region for bigger couplings and the non-uniqueness can be explained by a change in the non-unitarity or equivalently in the active-sterile mixing. For a fixed Majorana coupling the active-sterile mixing grows with increasing Dirac coupling. The observation of non-uniqueness is in accordance with eq. (4.45). If we look at areas of fixed colour we find almost straight lines where lines of different colours lie parallel to each other.

Considering the different boundaries we see that the effective mass sets the most important limits in the relevant energy scales. The bounds set by the branching ratio of the $\mu \to e\gamma$ decay becomes important at the TeV scale of the heavy neutrinos. Lepton universality does not play a role at all. These constraints all have in common that they set upper bounds to the non-unitarity or active-sterile mixing. The smaller the couplings the smaller is the maximally allowed active-sterile mixing.

Note that these boundaries cannot be expected to be exact as the process of averaging smears out these borders. Since the boundaries have been determined by fitting the outermost points of the prohibited regions, there are some viable points slightly below the border. As the mathematical shape of these borders is not known exactly, polynomials of convenient order have been used as fitting functions. Therefore, the shape of the borders especially at the end of the lines may not be taken as imperative. It is the approximate curvature and position of the lines that is of importance.

Note as well that all viable points lie well above the line which represents the ratio of 10^{-2} between Dirac and Majorana masses and can thus according to eq. (4.9) be trusted (see fig. 4.9).

The region of viable points is limited from the left by the value of minimal active-sterile mixing. This line is given by eq. (4.45), where $\mathscr{O}$ is set to $\mathbb{1}$, i.e. by

$$Y_D = i\frac{\sqrt{\langle\varphi\rangle}}{\langle H\rangle}\sqrt{Y_M}\sqrt{D_l}U^\dagger\,. \tag{4.51}$$

That means that in a logarithmic plot this relation is represented by a straight line with gradient 2.

At the top the region is limited by the requirement of the theory to be perturbative, i.e. by the limit of the Majorana coupling constant to be roughly below 1.

The general result obtained by this scan is that for the Meissner-Nicolai scenario Majorana and Dirac couplings can be approached upto a ratio of almost 10^{-2} within current experimental bounds. This happens at the cost of non-unitarity which is bounded from above by phenomenological requirements. Thus we do not have to introduce an arbitrary mass scale for the right-handed neutrinos and we can even bring the Dirac coupling upto approximately 10^{-2} at a Majorana coupling of order 1. In this case there is no reason to talk about unnaturally small Yukawa couplings.

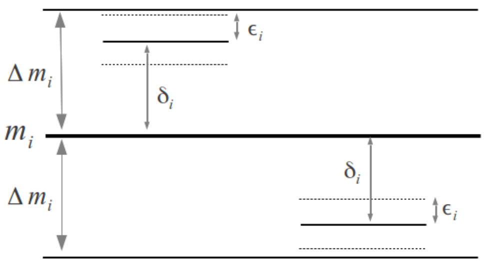


FIGURE 4.6: Small ϵ_i.

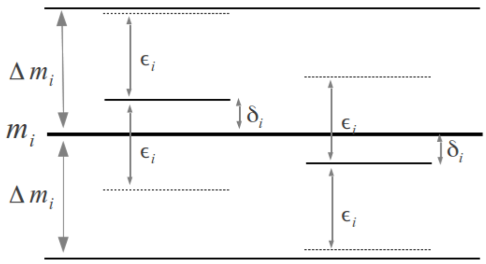


FIGURE 4.7: Big ϵ_i.

4.6 Parameter Scan for Pseudo-Dirac Neutrinos

In the case of Pseudo-Dirac neutrinos, i.e. if the right-handed Majorana scale is much smaller than the Dirac scale, we saw that we have to use a different parametrization. From eq. (4.31) we get

$$Y_D^2 = \frac{1}{\langle H \rangle^2} U \hat{m}_D^2 U^\dagger \,, \tag{4.52}$$

where $\hat{m}_D$ is the diagonal matrix with the central masses of neutrino pairs as entries and U is the PMNS matrix. This means that the Dirac Yukawa couplings do not depend on the small numbers ϵ_i which are now given by

$$\epsilon_i = (U_R^T Y_M \langle \varphi \rangle U_R)_{ii} \tag{4.53}$$

and thus not on the Majorana Yukawa coupling constants. As before we average over all entries of the Dirac and Majorana matrices. Therefore we should find a straight line at a discrete value of the averaged Dirac Yukawa couplings degenerated in the Majorana Yukawa coupling down to the pure Dirac neutrino case.

We now have to ask the question which are the experimental bounds imposed on Pseudo-Dirac neutrinos. Like seen before the right-handed masses lead to a small splitting of the squared masses of the Pseudo-Dirac pairs of $2\epsilon_i m_i$. This splitting, however, cannot be detected as long as it lies within the accuracy of the values for neutrino masses obtained from the measurement of mass squared differences within a certain hierarchy model. In dependency of how the scale of the Majorana couplings is chosen the central masses can be shifted more or less by a value δ without the actual mass eigenvalues exceeding or going below the accuracy limits of the neutrino masses (see fig. 4.6 and 4.7).

In our computation we therefore generated 50,000 random values for the right-handed masses where we roughly assumed the relation

$$\epsilon_i = M_{R,i} \,, \tag{4.54}$$

and for the deviations δ of the central masses. The ϵ_i were generated relative to the central mass by logarithmically generating a number between 10^{-6} and 1 and multiplying this number with the corresponding central mass, obtained from $\hat{m}_D$. The shift of the central mass is performed logarithmically as well within an interval of 50% below and above the assumed mass. These large deviations have been chosen to run well beyond the borders of phenomenological viability and thus to avoid the risk of influencing the border's shape by the choice of a possibly too small deviation interval. From that we then calculated the six mass eigenvalues and checked if they deviate by more than 5% from the assumed active neutrino masses. For the Pseudo-Dirac scenario we only investigated the Quasi-Degenerate Hierarchy mass spectrum. The assumed active neutrino masses are therefore given by eq. (4.50).

In fig. 4.8 we show the results of this scan where the colour in this case represents the effective electron neutrino mass. The viable region tapers at the top. This is something

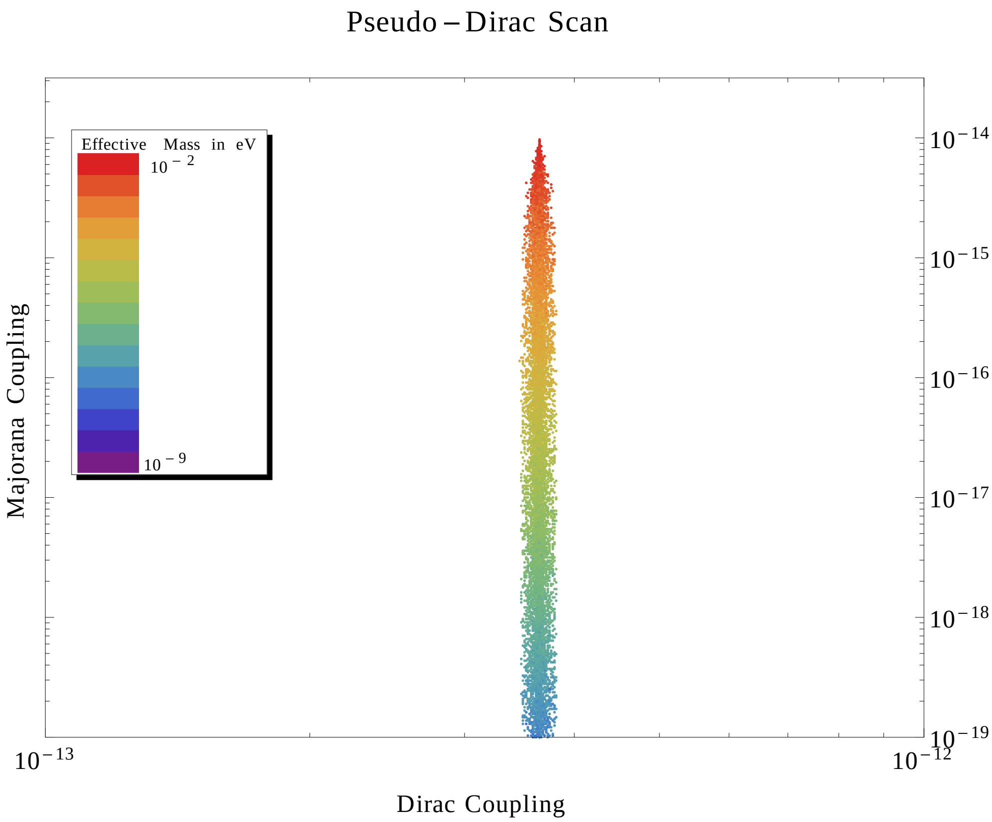


FIGURE 4.8: Parameter scan for Pseudo-Dirac neutrinos. The colour scheme represents the effective electron neutrino mass. The width of the viable region is determined by the current uncertainty of the active neutrino masses within the Quasi Degenerate Hierarchy.

we expected as with growing Majorana coupling the mass splitting of two neighboured mass eigenstates grows and the allowed deviation of the central mass has to decrease such that the splitting from the central value does not exceed or undercut the accuracy limit (see fig. 4.6 and 4.7). As soon as the right-handed couplings are under a certain limit they do not restrict the deviation of the central mass value and the width of the viable region becomes independent of the scale of the Majorana couplings.

The line has to go down to complete vanishing of the Majorana couplings which corresponds to the case of Dirac neutrinos. This, however, is not possible to be displayed in a logarithmic plot.

The colour scheme indicates that the effective electron neutrino mass is even for the values at the top one order of magnitude below the experimental bounds. Other bounds investigated for heavy sterile neutrinos do not play a role for Pseudo-Dirac neutrinos.

4.7 Summary of the Results

In the framework of conformal invariance we introduced 3 right-handed sterile neutrinos. The right-handed Majorana masses are most easily generated by introducing a singlet scalar which replaces the non-dynamical Majorana mass and equipping it with a vev. Within this model we found two regions in the parameter space of Dirac and Majorana Yukawa coupling constants that imply the correct phenomenology. These two regions are presented in fig. 4.9. The first region at Dirac couplings between 10^{-13} and 10^{-12} is called the Pseudo-Dirac region. It is a narrow band whose width is determined by the accuracy of the measured mass squared differences of the different neutrino mass states. The horizontal red line represents the limit of the effective electron neutrino mass given by 0.2 eV. For the Pseudo-Dirac case we find for the effective mass

$$|\langle m_{ee}\rangle| = |\sum_{i=1}^{6} m_i \boldsymbol{U}_{ei}| \approx \Delta m |\sum_{i=1}^{3} \boldsymbol{U}_{ei}| \approx \Delta m\,, \tag{4.55}$$

where Δm is the average Pseudo-Dirac splitting of the mass eigenstates. This splitting corresponds to the average right-handed Majorana mass. This means that the borderline for an effective mass of 0.2 eV lies at a Majorana Yukawa coupling of about $2 \cdot 10^{-13}$.
The second region between Dirac couplings of 10^{-9} and 10^{-1} is called the sub TeV region because the Majorana mass of the right-handed neutrinos is maximally of TeV scale. It is mainly limited from the right by the constraints on the effective electron neutrino mass obtained from measurements of the neutrinoless double beta decay. This boundary is represented by the red line. From the left it is limited by the boundary of smallest active-sterile mixing given by eq. (4.51) and is represented by the green line. Beyond that there are general constraints on the non-unitarity obtained from global fits. It is known that the non-unitarity has to be below a value of 10^{-2}. This boundary is represented by the brown line. As the non-unitarity is proportional to $RR^\dagger$ where R is given by eq. (4.46), we can deduce that in a logarithmic plot this boundary is given by a straight line with gradient 1. The line of smallest active-sterile mixing is given by a straight line of gradient 2. This leaves us with the triangular structure in the Yukawa space.
Finally it has to be emphasized that although this parameter scan has been explicitly performed for the Meissner-Nicolai model, the results can still be used for all theories yielding Majorana masses for the right-handed neutrinos. In this case the upper limit of the sub TeV region given by the requirement of the theory to be perturbative, is less compelling. Generally the triangle representing the sub TeV region can be continued to a Dirac coupling of 1.

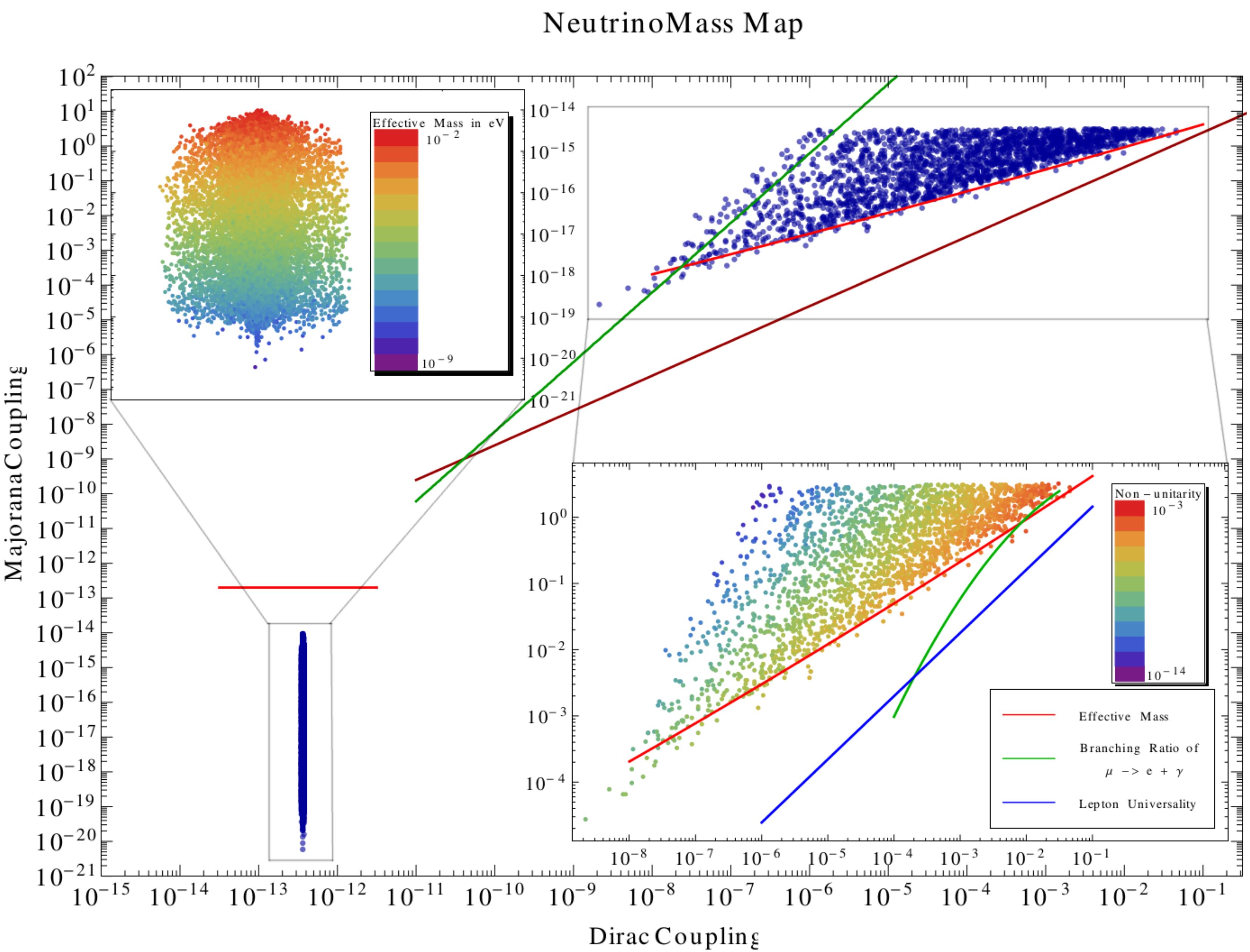


FIGURE 4.9: This plot shows the two viable regions in parameter space of Dirac and Majorana couplings for the Meissner-Nicolai model. The narrow band at a Dirac coupling between 10^{-12} and 10^{-13} is called the Pseudo-Dirac region, while the triangle between 10^{-9} and 10^{-1} is called the sub TeV region.

Chapter 5

Neutrino Mass Generation in Conformally Invariant Theories

5.1 Conformal Models of Neutrino Mass Generation within the SM Gauge Group

In this section we want to learn how extensions of the Standard Model could be performed in a conformally invariant way such that neutrino masses are generated. We will not yet extend the Standard Model gauge group but only study particle extensions taking the 6×6 mass matrix in the Majorana basis as a guidance. I.e. we assume the Standard Model to be at least extended by three right-handed neutrinos.
In the first subsection we will put together some topological building rules to see what is the basic shape of a conformal neutrino mass generation diagram. These rules will later help us to deduce more subtle topological lemmata for conformally invariant theories in subsection 3 and 4.
We will then study different possibilities to conformally influence different parts of the one-flavour 2×2 mass matrix taking non-conformal neutrino mass models as a guiding principle. Therefore, in the second subsection, we will study how to affect the left-handed Majorana entry of the mass matrix. This subsection is also supposed to show how conformal neutrino mass building works in general.
The fifth section will deal with models which influence the right-handed Majorana entry.

5.1.1 General Conformal Building Rules

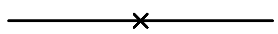
After studying the phenomenological requirements for the introduction of right-handed neutrinos we now want to see how we can build neutrino masses in conformally invariant theories from a diagrammatic point of view. In this section we want to show what are general topological building rules imposed by conformal invariance. The idea behind this diagrammatic approach is that we want to construct an effective Majorana mass term of the form

$$M_L \overline{\nu_L} \nu_L^c \,. \tag{5.1}$$

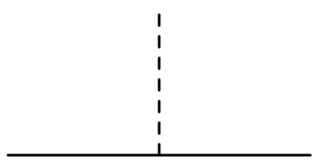
It is clear that this picture cannot cover the generation of pure Dirac neutrinos. Furthermore, we will generally use these diagrams to construct Majorana mass terms for other entries of the mass matrix in order to find possible corrections to the diagonal entries. It is part of the idea to integrate out several particles and it is important to make clear which particles are integrated out and which particles will stay in the theory.
Generally we will have an incoming particle of certain chirality, e.g. the left-handed neutrino ν_L and its antiparticle of opposite chirality as an outgoing particle, e.g. ν_L^c which is right-handed. In between we assume the fermions only to couple via Yukawa couplings of the form

$$\overline{\psi_L} \psi_R \varphi \;\; \text{and} \;\; \overline{\psi_R} \psi_L \varphi \,, \tag{5.2}$$

where the ψ are fermions and φ represents a scalar. Therefore, the coupling of the incoming fermion line results in an outgoing fermion line. As fermions cannot have a vev the result is a connected fermion line between incoming and outgoing fermion.
Mass terms are forbidden in the Lagrangian, i.e. no diagrams containing parts like

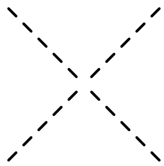


are allowed. Instead we need scalar insertions obtained from eq. (5.2) which are represented by the diagram

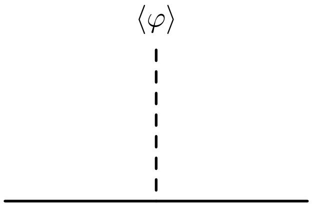


We will call these kind of insertions 'mass insertions'. Each diagram needs an odd number of mass insertions for the following reason. For the generation of Majorana masses we need an incoming particle and its outgoing antiparticle with opposite chirality. The Yukawa couplings given by eq. (5.2) interchange chirality of the fermion. Consequently we need an odd number of these couplings, i.e. mass insertions, such that incoming and outgoing fermion have opposite chirality.

For the scalars the following rules hold. Conformal invariance only allows for potential couplings which connect 4 scalars, i.e. diagrams of the form



Note that we work within the flavour basis, i.e. we use fields that appear in the unbroken Lagrangian. The breaking of the symmetry will be denoted by the insertion of a scalar vev which will look like



These most general rules will be used throughout this work and will later serve to derive more special topological rules with regard to certain questions.

5.1.2 Modifying the Left-Handed Majorana Entry

In this section we want to see which models influence the left-handed Majorana entry of the mass matrix. This task is special insofar as generating left-handed Majorana masses suffices to explain the currently known neutrino phenomenology. Thus we would not even need right-handed neutrinos. On the other hand, if they exist we can also generate an effective left-handed Majorana mass term by integrating them out. This was implicated in the Meissner-Nicolai model when we described the mass generation by a diagram. But there is a conceptual difference between the diagrammatic approach and calculating mass eigenvalues. In the former case we integrated out the right-handed neutrinos and find an effective Majorana mass term for the left-handed neutrinos, whereas in the latter case we build a Majorana neutrino out of left- and right-handed neutrinos such that the mass matrix is diagonal. These two pictures can only be compared if the active-sterile mixing is small enough. We will sometimes mix these two pictures assuming that this is the case. Therefore, we will sometimes not only affect the left-handed entry but also integrate the right-handed neutrinos out like we did in the Meissner-Nicolai model.

We will approach this subject by considering the non-conformal motivation and then finding conformal models which have tree-level diagrams as main contributions. Still we will add higher order terms to see what topologies are in principle admissible. From there we will then derive two topological lemmata, one dealing with radiative mass generation.

Non-Conformal Motivation

If we want to generate left-handed Majorana masses and do not necessarily want to introduce right-handed neutrinos we can ask how the scalar sector can be expanded such that left-handed neutrinos couple to those. We therefore couple the $SU(2)_L$ doublet containing the left-handed neutrino L to its antiparticle doublet defined by

$$L^c = \gamma_0 C i \sigma_2 L^* \,, \tag{5.3}$$

where $\gamma_0 C$ acts on the Dirac space like in chapter 3, whereas the σ_2 acts on the $SU(2)$ space. Doing this we find

$$\overline{L} L^c \sim (2,1) \times (2,1) = (1,2) + (3,2) \,, \tag{5.4}$$

where the brackets denote the quantum numbers $(SU(2)_L \times U(1)_Y)$. Thus the group theoretical analysis shows that L and L^c can be combined to a $SU(2)$ singlet or a $SU(2)$ triplet with hypercharge $Y = 2$. Therefore we can extend the scalar sector by either a triplet $\Delta : (3,-2)$ or by a singlet $\delta : (1,-2)$ to form a SM singlet from a Yukawa coupling. For both cases we will see examples how neutrino masses can be generated. In the case of the scalar triplet the most famous non-conformal mechanism is the type II seesaw mechanism [43–46], while for the extension by a scalar singlet the Zee-Babu model will be presented [47, 48]. We will see both models in slightly varied form when investigating conformally invariant models.

Beyond expanding the scalar sector or adding singlet right-handed neutrinos, there is a third way of extending the SM particle content that will lead to neutrino masses. This is we can introduce a fermionic triplet $\Sigma : (3,0)$. This particle content yields the type III seesaw mechanism which will be discussed as well [49–51].

- **Type II Seesaw Mechanism**

  Introducing a triplet scalar like described above yields the additional Yukawa term[1]

$$-\mathscr{L}_Y = g_\Delta \bar{L} \vec{\sigma} \Delta L^c + h.c. \,, \tag{5.5}$$

[1]Note that from now on we will only consider the one-flavour case.

where g_Δ is a coupling constant. We define the 2×2 matrix

$$\mathbf{\Delta} = \vec{\sigma}\Delta = \begin{pmatrix} \frac{1}{\sqrt{2}}\Delta_- & \Delta_0 \\ \Delta_{--} & \frac{1}{\sqrt{2}}\Delta_- \end{pmatrix}, \tag{5.6}$$

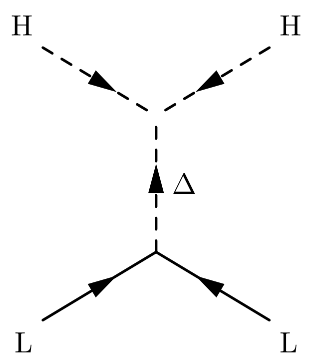
which transforms under the $SU(2)_L$ group like

$$\mathbf{\Delta} \to U\mathbf{\Delta}U^\dagger, \tag{5.7}$$

where U is the normal transformation matrix for $SU(2)$ doublets.
Beyond that there are additional terms in the potential involving the triplet. Relevant for the type II seesaw mechanism is the term

$$\mu[H^T i\sigma_2 \mathbf{\Delta} H + h.c.], \tag{5.8}$$

where μ is a coupling constant with mass dimension 1. This means that this term would not be allowed in a conformally invariant theory.
Using these couplings we can construct the diagram



If we then integrate out the scalar triplet, this diagram leads to the effective dimension 5 operator

$$\frac{1}{2\Lambda}\left(\bar{L}^c\tilde{H}^*\right)\left(\tilde{H}^\dagger L\right) + h.c., \tag{5.9}$$

which is called the Weinberg Operator [52]. Λ includes the mass of the scalar triplet and the corresponding couplings. Eq. (5.9) then yields after spontaneous symmetry breaking the left-handed Majorana mass term

$$-\mathscr{L}_m = \frac{1}{2}M_L\overline{\nu_L}\nu_L^c + h.c., \tag{5.10}$$

where M_L is given by

$$M_L = \frac{\langle H\rangle^2}{\Lambda^2}\mu. \tag{5.11}$$

We see that the bigger the mass scale of the triplet the smaller the neutrino mass. In this way the mass of the triplet suppresses the mass of the neutrinos which explains the name seesaw mechanism.

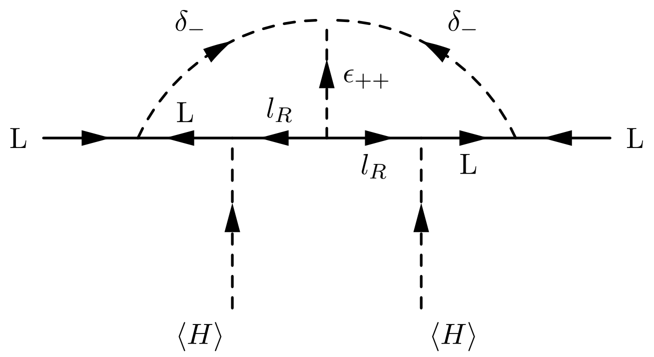
- **The Zee-Babu Model**
The Zee-Babu model introduces a singlet scalar δ_- which has a single negative electric charge and a doubly charged singlet scalar ϵ_{++}. The new part of the Yukawa Lagrangian induced by these scalars is given by

$$-\mathscr{L}_Y = g_\delta \bar{L} L^c \delta_- + g_\epsilon \overline{l_R^c} l_R \epsilon_{++} + \overline{L} H l_R + h.c. \,, \tag{5.12}$$

where l_R is the charged right-handed lepton. The relevant extension of the potential is given by the term

$$\mu \delta_- \delta_- \epsilon_{++} + h.c. \,, \tag{5.13}$$

where μ is a coupling constant with mass dimension 1. With these couplings we can construct the following diagram:



In the Zee-Babu model the left-handed Majorana masses are generated radiatively. It has the advantage that for each loop the mass is suppressed by a factor of $1/(16\pi^2) \approx 10^{-2}$ and could thus give a very natural explanation for the smallness of neutrino masses. Evaluating this diagram yields the neutrino mass

$$M_L = 8\mu m_l^2 g_\delta^2 g_\epsilon I \,, \tag{5.14}$$

where m_l is the mass of the charged lepton and

$$\begin{aligned} I = \int \frac{\mathrm{d}^4 p}{(2\pi)^4} \int \frac{\mathrm{d}^4 q}{(2\pi)^4} \frac{1}{p^2 - m_l^2} \frac{1}{q^2 - m_l^2} \\ \frac{1}{p^2 - m_\delta^2} \frac{1}{q^2 - m_\delta^2} \frac{1}{(p-q)^2 - m_\epsilon^2} \,. \end{aligned} \tag{5.15}$$

- **Type III Seesaw Mechanism**

  Expanding the SM particle content by a fermionic triplet yields the following additional terms in the mass Lagrangian

$$- \mathscr{L}_m = M_\Sigma \, \mathrm{Tr} \left[\overline{\Sigma^c} \Sigma \right] + h.c. \tag{5.16}$$

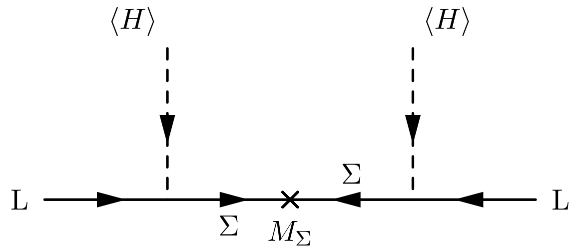
and in the Yukawa Lagrangian

$$- \mathscr{L}_Y = g_\Sigma \tilde{H}^\dagger \overline{\Sigma} L + h.c. \,, \tag{5.17}$$

where in this case the fermionic triplet is already in its 2×2 representation

$$\Sigma = \begin{pmatrix} \frac{1}{\sqrt{2}} \Sigma_0 & \Sigma_+ \\ \Sigma_- & \frac{1}{\sqrt{2}} \Sigma_0 \end{pmatrix} \tag{5.18}$$

and it transforms under the $SU(2)$ gauge group exactly the way the 2×2 representation of the scalar triplet does. Note as well that M_Σ and g_Σ are pure numbers. Generally these two objects could be matrices, but for simplicity we do not consider such a case. With these new terms we can build the diagram



In this case the neutrino mass is proportional to the inverse mass of the fermionic triplet, i.e.

$$M_L \propto \frac{\langle H \rangle^2}{M_\Sigma} \,. \tag{5.19}$$

Thus for the same reason, namely that the neutrino mass is suppressed by the large mass of a different particle, this is called seesaw mechanism.

Conformal Tree-Level Models

With the different particle contents suggested by the non-conformal analysis we can now look for conformally invariant models with similar particle content. It will be very important in this section to point out which particles have been integrated out and which picture of neutrino mass generation we are considering. It will become clear in this section how the two pictures explained before are related to each other. To do so and for the sake of completeness we will come back to the already known Meissner-Nicolai model. Note again that for the rest of this work we will only consider the one-flavour case.

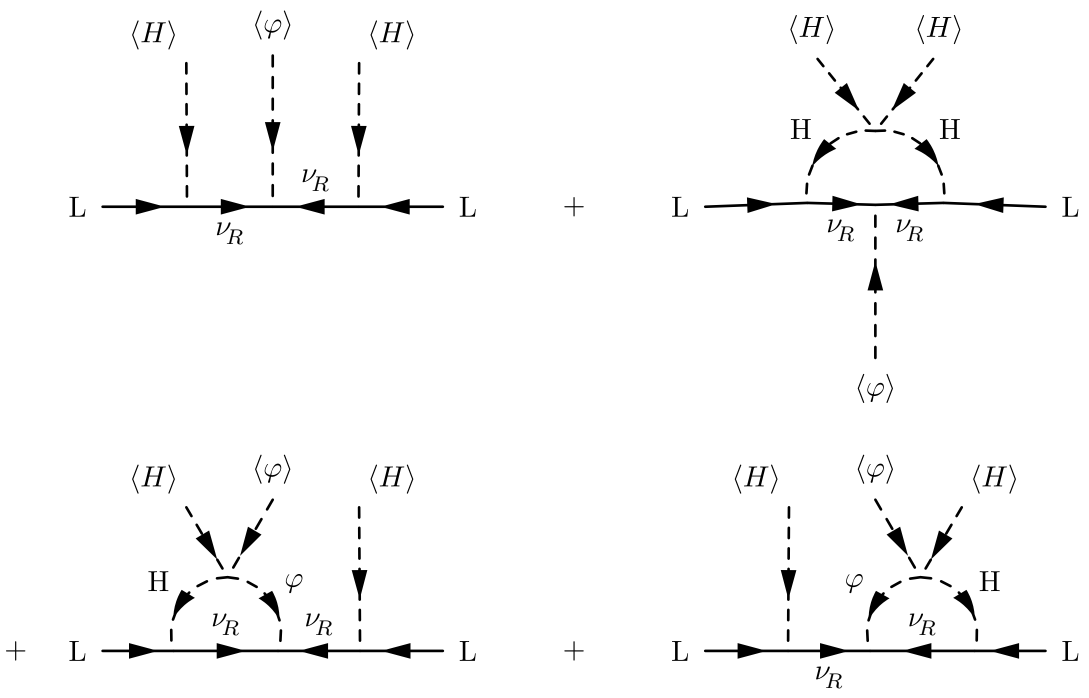
- **SM + $\boldsymbol{\nu_R + \varphi}$**

  Particle content: $L : (2, -1)$; $H : (2, 1)$; $\nu_R : (1, 0)$; $\varphi : (1, 0)$,

  Yukawa Lagrangian: $-\mathscr{L}_Y = g_H \overline{L}\tilde{H}\nu_R + \frac{1}{2} g_\varphi \varphi \overline{\nu_R^c}\nu_R + \text{h.c.}$

  Potential: $V_{\text{I}} = \lambda_H (H^\dagger H)^2 + \lambda_\varphi (\varphi^\dagger \varphi)^2 + \lambda_{H\varphi} (\varphi^\dagger \varphi)(H^\dagger H)$

  With this we find the diagrams



We have already seen the first diagram while the other three are new. They are one-loop corrections to the first diagram and have thus a smaller contribution to the total neutrino mass. Further contributions have either at least two loops or 9 mass insertions. However, these diagrams have even smaller contributions.

It is now important to note that by integrating out the right-handed neutrinos we reduce the 2×2 matrix to a 1×1 matrix, i.e. to a real number. In this way it is not only a correction to the left-handed Majorana mass term in the 2×2 matrix but rather the reduction of the whole matrix.

We therefore want to consider now a real correction of the left-handed Majorana entry without integrating out the right-handed neutrinos. We do this by introducing a scalar triplet.

- **SM + $\boldsymbol{\Delta}$**

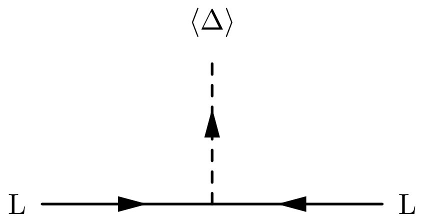
  Particle content: $L : (2, -1);\ H : (2, 1);\ \Delta : (3, -2)$

  Yukawa Lagrangian: $-\mathscr{L}_Y = g_\Delta \bar{L}\vec{\sigma}\Delta L^c + h.c. = g_\Delta(\bar{L}\vec{\sigma}\Delta L^c + \bar{L}^c\vec{\sigma}\Delta^* L)$

  Potential:

$$V_{\text{II.0}} = \lambda_H (H^\dagger H)^2 + \lambda_{\boldsymbol{\Delta}T} \operatorname{Tr}(\boldsymbol{\Delta}^\dagger \boldsymbol{\Delta})^2 + \lambda_{T\boldsymbol{\Delta}} (\operatorname{Tr} \boldsymbol{\Delta}^\dagger \boldsymbol{\Delta})^2 + \lambda_{H\boldsymbol{\Delta},1} (H^\dagger H) \operatorname{Tr} \boldsymbol{\Delta}^\dagger \boldsymbol{\Delta} + \lambda_{H\boldsymbol{\Delta},2} H^\dagger \boldsymbol{\Delta}\boldsymbol{\Delta}^\dagger H\,,$$

All diagrams with upto 3 mass insertions and one loop are given by



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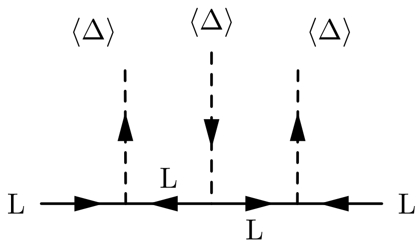

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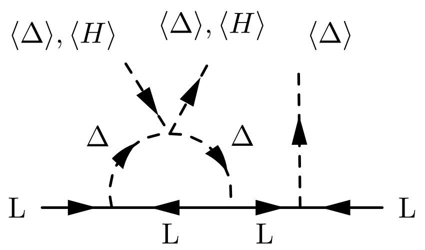
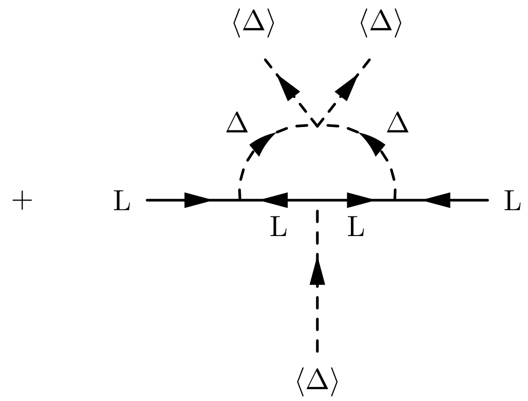


+ same diagram closing the loop between 2nd and 3rd insertion



If we consider the second, third and fourth diagram, we note that these are not 1-Particle-Irreducible (1PI) diagrams. I.e. we include radiative corrections to the external line which we should not do when calculating mass corrections. Thus these kind of diagrams have to be dismissed. However, as our way of finding all relevant diagrams up to a certain number of mass insertions and involved loops is based on a topological approach, we will further display these diagrams but mention if they have to be dismissed. Consequently in this theory only the first and the fifth diagram are viable.
The first diagram yields a left-handed Majorana mass term given by

$$-\mathscr{L}_m = \frac{1}{2} M_L \overline{\nu_L^c} \nu_L \,, \tag{5.20}$$

where

$$M_L = 2 g_\Delta \langle \Delta_0 \rangle \,. \tag{5.21}$$

Note that the vev $\langle \Delta_0 \rangle$ couples to the W- and Z-bosons and thus influences its masses. Experimental bounds on the 'ρ-parameter', which is in this case given by

$$\rho = \frac{1 + 2\langle H \rangle^2 / \langle \Delta_0 \rangle^2}{1 + 4\langle H \rangle^2 / \langle \Delta_0 \rangle^2} \,, \tag{5.22}$$

require the ratio

$$\frac{\langle \Delta_0 \rangle}{\langle H \rangle} < 0.07 \,. \tag{5.23}$$

In contrast to non-confromal theories the potential term

$$\mu [H^T i \sigma_2 \mathbf{\Delta} H + h.c.] \,, \tag{5.24}$$

is forbidden as it would spoil conformal invariance. Therefore, if we assume that

there are no right-handed neutrinos in the theory, the neutrino mass is not affected by the Higgs vev but only by the triplet vev. Therefore, in this conformally invariant theory the smallness of the neutrino masses could be explained by the smallness of the triplet vev, which is, as mentioned, phenomenologically implied.
If, however, the triplet is the only extension of the scalar particle content, then this conformally invariant theory is not able to explain the scale of the Higgs mass as the correction due to the triplet vev is too small. Consequently this theory alone is phenomenologically not viable. The situation changes if we additionally introduce a singlet scalar φ.

- **SM + $\boldsymbol{\Delta}$ + $\boldsymbol{\varphi}$**

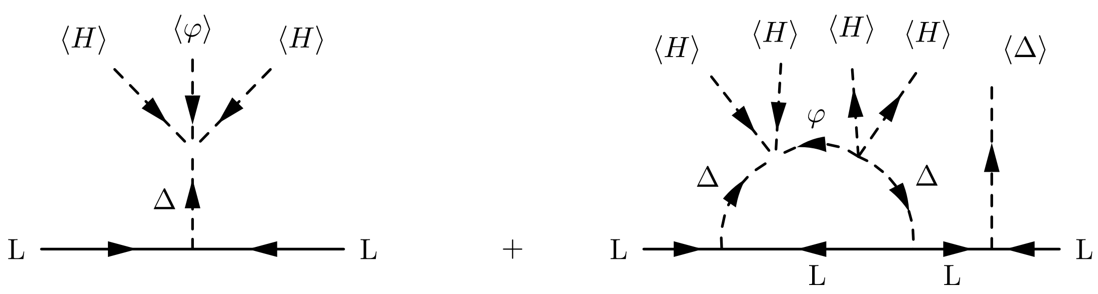
  Particle content: $L : (2, -1)$; $H : (2, 1)$; $\Delta : (3, -2)$; $\varphi : (1, 0)$

  Yukawa Lagrangian: $-\mathcal{L}_Y = g_\Delta \bar{L}\vec{\sigma}\Delta L^c + h.c. = g_\Delta(\bar{L}\vec{\sigma}\Delta L^c + \bar{L}^c\vec{\sigma}\Delta^* L)$

  Potential:

$$V_{\text{II.1}} = V_{\text{II.0}} + \lambda_\varphi(\varphi^\dagger\varphi)^2 + \lambda_{H\varphi}(\varphi^\dagger\varphi)(H^\dagger H) + \lambda_{\varphi\boldsymbol{\Delta}}(\varphi^\dagger\varphi)\,\mathrm{Tr}\,\boldsymbol{\Delta}^\dagger\boldsymbol{\Delta} + \lambda_{\varphi\boldsymbol{\Delta}H}[\varphi H^T i\sigma_2\boldsymbol{\Delta}H + h.c.]$$

This theory yields the following diagrams in addition to those of the previous theory:



These are all diagrams with upto three mass insertions and one loop except for a third diagram that is identical to the third diagram but closes the loop between second and third mass insertion. Like before the second diagram is topologically possible but has to be dismissed as it is not a 1PI diagram.
The theory at hand is the conformal analogon of the type II seesaw mechanism. If we forbid the vev $\langle\Delta_0\rangle$, than the main contribution comes from the first diagram which yields the neutrino mass

$$M_L = 2g_\Delta \frac{\lambda_{\varphi\Delta H}}{M_\Delta^2}\langle\varphi\rangle\langle H\rangle^2\,, \tag{5.25}$$

where M_Δ is the physical mass of the scalar triplet.

This theory influences only the left-handed Majorana entry without integrating out the right-handed neutrinos. Thus if we further consider the theory to have right-handed neutrinos we get back the full mass matrix

$$\mathcal{M} = \begin{pmatrix} M_L & m_D \\ m_D & M_R \end{pmatrix} . \tag{5.26}$$

Like shown we can then find the eigenvalues and the eigenbasis of this matrix. On the other hand we can move into the picture where we integrate the right-handed neutrinos out.

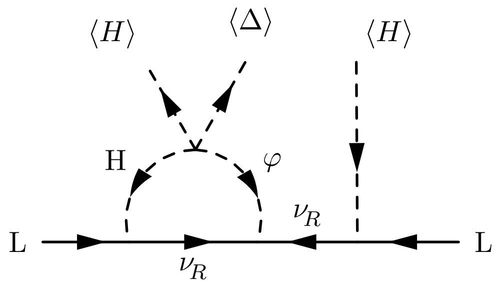
- **SM+$\boldsymbol{\nu_R}$+$\boldsymbol{\varphi}$ +$\boldsymbol{\Delta}$**

Particle content: $L : (2, -1);\ H : (2, 1);\ \Delta : (3, -2);\ \varphi : (1, 0);\ \nu_R : (1, 0)$

Yukawa Lagrangian: $-\mathscr{L}_Y = g_H \bar{L}\tilde{H}\nu_R + g_\varphi \varphi \bar{\nu}_R^c \nu_R + g_\Delta \bar{L}\vec{\sigma}\Delta L^c + h.c.$

Potential: $V = V_{\mathrm{II.1}}$

The following two diagrams are additional to those of the 'SM + ν_R + φ - theory' and the 'SM + Δ + φ - theory':



+ same diagram where the loop is closed between φ and right H

We saw that two left-handed $SU(2)$ doublets can be combined to a triplet or a singlet. Another possibility is therefore to couple this singlet combination to a scalar $SU(2)$ singlet.

- **SM + $\boldsymbol{\delta_-}$**

Particle content: $L : (2, -1);\ H : (2, 1);\ \delta_- : (1, -2)$

Yukawa Lagrangian: $-\mathscr{L}_Y = g_\delta \bar{L}L^c\delta_- + h.c. = g_\delta(\bar{L}L^c\delta_- + \bar{L}^c L\delta_-^\dagger)$

Potential:

$$V_{\mathrm{III}} = \lambda_H(H^\dagger H)^2 + \lambda_\delta(\delta_-^\dagger\delta_-)^2 + \lambda_{H\delta}(H^\dagger H)(\delta_-^\dagger\delta_-)$$

This theory alone cannot generate a left-handed Majorana mass as the singlet scalar has electric charge -1. We need an odd number of mass insertions and thus, as the δ is our only connection to the fermion line, we need an odd number of δ insertions. Consequently there has to be a net electric charge flowing into or out of the fermion line. As we have a neutrino, i.e. a neutral particle, as the incoming and a neutrino as the outgoing particle, however, there may not be a net electric charge flow from or to the fermion line. This contradiction prevails as long as there is not an additional scalar with a charged component that couples to the fermion line. A different argumentation is based on the fact that conformally invariant theories containing upto $SU(2)$ triplets need a triplet or a singlet scalar vev to yield neutrino masses, which will be seen in the next section (Lemma 1.1). Hence, as δ can not gain a vev and as there are no further singlet or triplet scalars in the theory, neutrino masses cannot be generated.
Therefore, we extend this theory by the scalar triplet examined before.

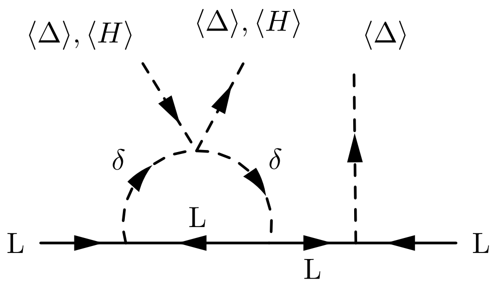
- **SM + $\boldsymbol{\Delta}$ + $\boldsymbol{\delta_-}$**

  Particle content: $L:(2,-1);\ H:(2,1);\ \Delta:(3,-2);\ \delta_-:(1,-2)$

  Yukawa Lagrangian: $-\mathscr{L}_Y = g_\Delta \bar{L}\vec{\sigma}\Delta L^c + g_\delta \bar{L}L^c\delta_- + h.c.$

  Potential: $V = V_{\text{II.0}} + V_{\text{III}} + \lambda_{\delta\boldsymbol{\Delta}}(\delta_-^\dagger\delta_-)\,\text{Tr}\,\boldsymbol{\Delta}^\dagger\boldsymbol{\Delta}$

  The diagrams we get in addition to those of the 'SM + Δ '- theory are given by



\+ same diagram reflected on a vertical axes.

Unfortunately both diagrams have to be dismissed for the same reason like other diagrams above. This theory can also be extended by an uncharged singlet scalar and we can integrate out the right-handed neutrinos. This, however, would be beyond the scope.
Like seen in the non-conformal case it is also possible to introduce a triplet fermion to couple to the left-handed doublet. Unlike in the non-conformal scenario we now have to introduce an uncharged singlet scalar to generate neutrino masses.

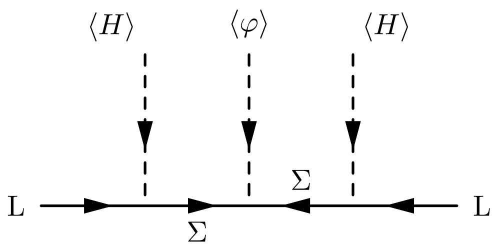
- **SM + $\Sigma + \varphi$**

  Particle content: $L:(2,-1);\ H:(2,1);\ \Sigma:(3,0);\ \varphi:(1,0)$,

  Yukawa Lagrangian: $-\mathscr{L}_Y = g_\Sigma \tilde{H}^\dagger \overline{\Sigma} L + g_\varphi \varphi \operatorname{Tr}\left[\overline{\Sigma^c}\Sigma\right] + h.c.$

  Potential: $V = V_{\rm I}$

  The main contribution to the neutrino mass is given by



This diagram yields the mass

$$M_L = g_\Sigma^2 \frac{\langle H\rangle^2}{g_\varphi \langle\varphi\rangle}\,. \tag{5.27}$$

5.1.3 Topological Lemma 1 - Weinberg Operator

After having investigated several models we observe that all diagrams so far involve at least one vacuum expectation value other than the Higgs vev. We will argue that this is not a coincidence but rather a topological necessity of conformally invariant theories including upto $SU(2)$ triplets.

To prove this we first note that any diagram has an even number of doublet scalar mass insertions. This is because all diagrams generating left-handed Majorana masses have the left-handed doublet as the incoming and the outgoing particle, i.e. we have to start and end up with a doublet. If we assume that the theory has only upto $SU(2)$ triplet scalars and fermions, the only possibilities to connect two fermionic doublets are Yukawa couplings with a scalar triplet or singlet. Connecting a doublet fermion to a singlet fermion involves a doublet scalar. Equivalently a doublet and a triplet fermion are connected via a scalar doublet. Furthermore, two fermion singlets connect to a singlet scalar, two fermion triplets to a singlet scalar as well and a triplet and singlet fermion to a triplet scalar (see Table 5.1). Thus scalar doublets occur if and only if we connect a fermionic doublet to a fermionic non-doublet. Therefore in order to start and end up with a fermion doublet we necessarily have an even number of scalar doublet mass insertions.

Secondly note that in any theory including upto $SU(2)$ triplets there are only potential

	S	D	T
S	$\varphi\overline{S}S$	$\overline{D}\tilde{\phi}S$	$\mathrm{Tr}\,[\overline{T}\Delta S]$
D		$\overline{D}D^c\varphi$, $\overline{D}\mathbf{\Delta}D^c$	$\tilde{\phi}^\dagger\overline{T}L$
T			$\mathrm{Tr}\,[\varphi\overline{T^c}T]$

TABLE 5.1: Possible dimension 4 Yukawa coupling terms. S, D and T denote singlet, doublet and triplet fermions respectively. φ, ϕ and Δ denote singlet, doublet and triplet scalars respectively.

couplings possible that involve an even number of $SU(2)$ doublets. Therefore, each doublet line will couple to an odd number of doublet lines. As the product of an even and an odd number is an even number there will be left an even number of doublet lines. Connecting some of these lines and producing a loop will not change this fact as this closing reduces the number of external doublet lines by an even number.

On the other hand two fundamental building rules for conformally invariant neutrino mass generation say that firstly there is always an odd number of mass insertions and secondly potential couplings always connect four lines. Both together yield that there has to be left an odd number of scalar external lines. Consequently as there has to be an odd number of vevs but an even number of doublet vevs, there has to remain a singlet or a triplet vev. Note, however, that this proof is based on the assumption that there are no fermion or gauge boson loops involved. This finding can be summarized in two lemmata.

Lemma 1.1: *If there are no gauge boson or fermion loops possible, a conformally invariant theory with upto $SU(2)$ triplet scalars and fermions needs a singlet or triplet scalar vacuum expectation value to generate left-handed Majorana neutrino masses.*

Lemma 1.2: *In a conformally invariant theory with upto $SU(2)$ triplet scalars and fermions and without fermion or gauge boson loops, left-handed Majorana neutrino masses cannot be generated via Weinberg's dimension 5 operator.*

The second lemma is true as Weinberg's dimension 5 operator generates left-handed Majorana neutrino masses by only the Higgs doublet getting a vev.

5.1.4 Topological Lemma 2 - Radiative Models

In this section we deal with the question if it is possible to choose the particle content and the vev structure of a theory such that the lowest order contribution to the left-handed

Majorana masses is fully radiative, i.e. that all scalar lines serving as mass insertions to the fermion line have to be connected in a loop. This is an attractive aim because, as mentioned before, every loop suppresses the contribution of a diagram by the kinematic factor of $1/(16\pi^2)$.

To guarantee this we first have to forbid vacuum expectation values for all scalars which can generally connect to the fermion line. We then have to examine under which circumstances these scalars can be coupled in loops in such a way that all final vev insertions are scalars different from those coupling to the fermion line.

Assume first that there may be no fermion or gauge boson loops and that the particle content is chosen such that the potential has only terms coupling particles pairwise, i.e. coupling two of the same particles to a total singlet and coupling this to another singlet produced in the same way. In this case all scalars connected to the fermion line can only be coupled in such a way that they either produce one scalar of the own kind and two of another or couple to a particle of the own kind coming from the fermion line and thus reducing the number of its species by an even number. So either the number of a species stays the same or reduces by an even number. As there has to be an odd number of mass insertions to the fermion line it is thus impossible to combine all scalars connected to the fermion line in a loop without producing at least one external line that already couples to the fermion line.

We have to note that there is always a scalar that can couple to the fermion line namely the Higgs doublet connecting left-handed and right-handed charged lepton. We will, however, consider these kind of mass insertions simply as the masses of the charged leptons. As there has to be an even number of doublet insertions, an odd number of total mass insertions and thus an odd number of non-doublet insertions the argumentation from above stays unaffected. We can summarize this result in a lemma.

Lemma 2: *In a conformally invariant theory without fermion or gauge boson loops it is impossible to generate left-handed Majorana neutrino masses in a fully radiative way if the potential contains only terms coupling scalars pairwise.*

Consequently there are five possibilities left that might circumvent Lemma 2 and ultimately yield fully radiative left-handed neutrino masses.

- **Possibility 1**: We can introduce a potential coupling of four different $SU(2)$ singlet scalars such that their hypercharges add up to zero. In this case one $SU(2)$ singlet with vanishing hypercharge has to be included as we need an electrically neutral scalar to gain a vev.

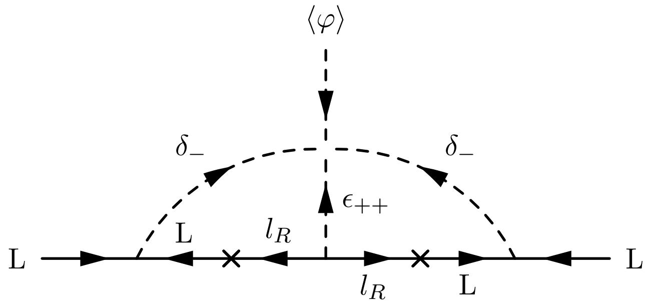
  With this kind of coupling it is indeed possible to construct a theory that generates neutrino masses fully radiatively. Consider as an example the following theory which includes the right-handed charged lepton l_R and is extended by the singly charged scalar δ_- and the doubly charged scalar ϵ_{++}:

Particle content: $L : (2, -1);\ l_R : (1, -2)$
$H_1 : (2, 1);\ \delta_- : (1, -2);\ \epsilon_{++} : (1, 4)$

Yukawa Lagrangian: $-\mathscr{L}_Y = g_\delta \bar{L} L^c \delta_- + g_\epsilon \overline{l^c_R} l_R \epsilon_{++} + \overline{L} H l_R + h.c.$

Potential: $V = \lambda \varphi \delta_- \delta_- \epsilon_{++} + h.c. + ...$

For this theory we find the radiative generation of neutrino masses represented by the diagram



The crosses denote the insertion of a Higgs vev, i.e. they represent the mass of the charged lepton. This theory is the conformally invariant analogon to the Zee-Babu model. The corresponding left-handed neutrino mass is given by

$$M_L = 8\lambda \langle \varphi \rangle m_l^2 g_\delta^2 g_\epsilon I \,, \tag{5.28}$$

where I is given by eq. (5.15).

- **Possibility 2**: We can introduce a potential coupling of 4 different $SU(2)$ doublets such that their hypercharges add up to zero in the following structure

$$\left(\phi_1^\dagger \phi_2\right) \left(\phi_3^\dagger \phi_4\right) \,. \tag{5.29}$$

However, this term alone cannot change the situation described above as there is always an even number of doublet insertions leaving an odd number of singlet and triplet insertions. The argumentation of pairwise potential couplings can now be used for the triplet and singlet mass insertions without being affected by this doublet term.

- **Possibility 3**: A potential term coupling 4 different $SU(2)$ triplets such that their hypercharges add up to zero in the following way

$$\left(\Delta_1^\dagger \Delta_2\right) \left(\Delta_3^\dagger \Delta_4\right) \,. \tag{5.30}$$

It is beyond the scope of this work to show an explicit example for the radiative generation of neutrino masses using a potential term like this.

- **Possibility 4**: A further term that can be introduced is given by the coupling

$$\varphi H_i^T i\sigma_2 \mathbf{\Delta}^\dagger H_j \,, \tag{5.31}$$

where φ is a $SU(2)$ singlet, H_i and H_j are doublets and Δ is a $SU(2)$ triplet with hypercharges such that they add up to zero in this term. That with the help of such a coupling the fully radiative generation of neutrino masses is possible can be seen in the following theory:

Particle content: $L_1 : (2, -1);\ L_2 : (2, -3);\ L_3 : (2, 0)$

$\Delta_1 : (3, -4);\ \Delta_2 : (3, -3);\ \Delta_3 : (3, -1)$

$H_1 : (2, 1);\ H_2 : (2, -3);\ H_3 : (2, 0)$

$\varphi : (1, 0)$

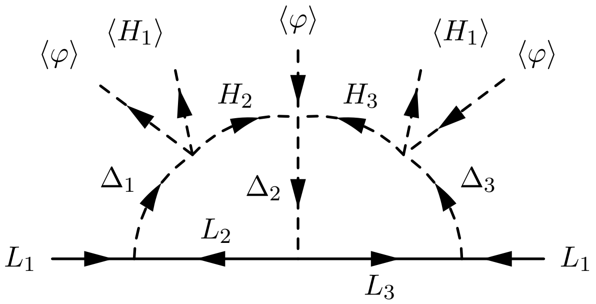
Note that all L are fermions and all Δ, H and φ are scalars.

Yukawa Lagrangian: $-\mathcal{L}_Y = g_a \bar{L}_1 \vec{\sigma} \Delta_1 L_2^c + g_b \bar{L}_2 \vec{\sigma} \Delta_2 L_3^c + g_c \bar{L}_3 \vec{\sigma} \Delta_3 L_1^c + h.c.$

Potential:

$$V = \left[\lambda_a \varphi \tilde{H}_1^T i\sigma_2 \mathbf{\Delta}_1^\dagger H_2 + \lambda_b \varphi H_2^T i\sigma_2 \mathbf{\Delta}_2^\dagger H_3 + \lambda_c \varphi H_3^T i\sigma_2 \mathbf{\Delta}_3^\dagger \tilde{H}_1 + h.c. \right] + ...$$

If we forbid the vevs $\langle \Delta_1 \rangle$, $\langle \Delta_2 \rangle$, $\langle \Delta_3 \rangle$, $\langle H_2 \rangle$ and $\langle H_3 \rangle$, then the following diagram describes the radiative generation of neutrino masses:



Admittedly this theory is phenomenologically problematic. But it is intended to show that it is possible to generate neutrino masses fully radiatively from a topological and gauge invariant point of view.

- **Possibility 5**: Furthermore we can allow fermion loops. However, an explicit example will not be presented.

5.1.5 Modifying the Right-Handed Majorana Entry

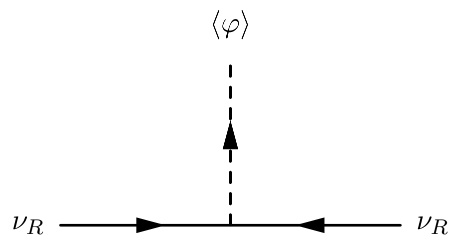
After having studied how the left-handed Majorana entries of the mass matrix $\mathcal{M}$ can be generated, we are now going to investigate how the right-handed Majorana entries, i.e. the right-handed neutrino masses can be produced. In the previous section we sometimes integrated out the right-handed neutrinos. In this picture the corrections to the right-handed neutrino masses would be collected in diagrams with a higher number of mass insertions. We already saw one way how to generate right-handed masses in a conformally invariant theory. By introducing a total singlet scalar φ we obtained the term

$$g\varphi\overline{\nu_R}\nu_R^c + h.c.\,, \tag{5.32}$$

which after spontaneous symmetry breaking leads to the right-handed mass

$$M_R = 2g\langle\varphi\rangle\,, \tag{5.33}$$

which is represented by the diagram



There are, however, further ways to influence the right-handed Majorana mass. The first possibility we want to study is to introduce a scalar and a fermion triplet and a scalar singlet.

Introduction of a Scalar and a Fermion Triplet

Particle content: $\nu_R : (1,0);\ \Sigma : (3,0);\ H : (2,1);\ \Delta : (3,0);\ \varphi : (1,0)$

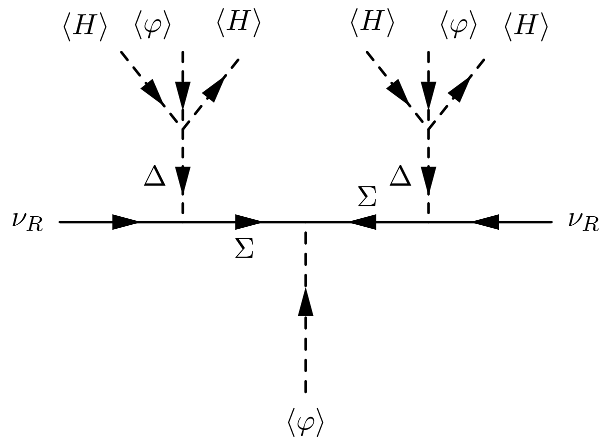
With this particle content we can write down the following terms:

Yukawa Lagrangian: $-\mathscr{L}_Y = g_\Delta \operatorname{Tr}[\overline{\Sigma}\boldsymbol{\Delta}\nu_R] + g_{\varphi,1}\operatorname{Tr}[\varphi\overline{\Sigma^c}\Sigma] + g_{\varphi,2}\varphi\overline{\nu_R}\nu_R^c + h.c.$

The relevant potential coupling that can be written down is given by

Potential: $V = \lambda\varphi H^T i\sigma_2 \boldsymbol{\Delta}^\dagger \tilde{H} + h.c.$

Furthermore we forbid the vev of Δ. In addition to the diagram obtained from eq. (5.32) we get the diagram



Note that the scalar triplet Δ cannot be used to generate left-handed Majorana masses as it has the wrong hypercharge. From both diagrams the right-handed mass is found to be given by

$$\begin{aligned} M_R &= 2g_{\varphi,2}\langle\varphi\rangle + 2\lambda^2 g_\Delta^2 \frac{\langle H\rangle^4\langle\varphi\rangle^2}{2g_{\varphi,1}\langle\varphi\rangle \cdot M_\Delta^4} \\ &= \left(2g_{\varphi,2} + \frac{\lambda^2 g_\Delta^2}{g_{\varphi,1}}\left(\frac{\langle H\rangle}{M_\Delta}\right)^4\right)\langle\varphi\rangle \end{aligned}, \tag{5.34}$$

where M_Δ is the physical mass of Δ after symmetry breaking. We see that there is a lot of potential in the second term to make the right-handed scale very big. The factor $\left(\frac{\langle H\rangle}{M_\Delta}\right)^4$ will lift the scale of M_R 4 orders of magnitude above the electroweak scale if we make the reasonable assumptions that all couplings are of order 1, the vev of φ is of electroweak scale and the scale of M_Δ is one order of magnitude below the one of $\langle H\rangle$. This lift in M_R allows for an increased scale of the Dirac masses in a seesaw type I scenario. Within the picture of integrating out the right-handed neutrinos the diagram above would be represented by a diagram with 5 mass insertions.

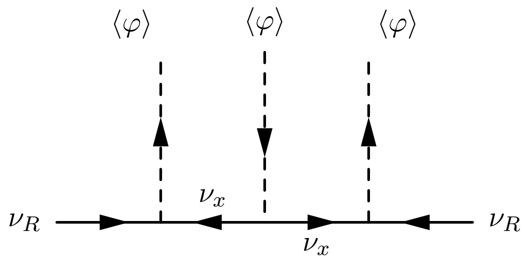
Introduction of a Further Sterile Neutrino

A further possibility to influence the right-handed neutrino masses is to introduce another sterile neutrino.[2] We choose this one to be right-handed as well. Furthermore, we introduce a singlet scalar to get the following relevant terms in the Yukawa Lagrangian:

$$-\mathscr{L}_Y = g_{RR}\varphi\overline{\nu_R}\nu_R^c + g_{xx}\varphi\overline{\nu_x}\nu_x^c + g_{Rx}\varphi\overline{\nu_R}\nu_x^c + h.c. \tag{5.35}$$

Besides the diagram induced by eq. (5.32) this theory yields the diagram

[2] Note that we work with one flavour only. Therefore we add only one further sterile neutrino and not three.



Both diagrams together yield the mass

$$\begin{aligned} M_R &= 2g_{RR}\langle\varphi\rangle + 2g_{Rx}^2 \frac{\langle\varphi\rangle^2}{2g_{xx}\langle\varphi\rangle} \\ &= 2\langle\varphi\rangle \left(g_{RR} + \frac{g_{Rx}^2}{g_{xx}} \right) \end{aligned} . \tag{5.36}$$

We see that both diagrams have similar contributions. Furthermore there is no way to distinguish between ν_R and ν_x but by their masses. In other words ν_R and ν_x are different only in their flavour. By changing from the flavour to the mass basis of the right-handed neutrinos the diagram above becomes obsolete.

The situation becomes much more interesting when we introduce a new symmetry that distinguishes both particles.

5.2 Conformal Models of Neutrino Mass Generation with an Additional Hidden Sector Symmetry

In the last chapter we studied conformally invariant theories that yield neutrino masses within the SM gauge group. As an orientation we used the one-flavour 2×2 mass matrix in the basis (ν_L, ν_R^c), where we also sometimes considered the picture of integrated out right-handed neutrinos. At the end of this chapter we investigated how the right-handed Majorana entry of the 2×2 matrix can be affected. One possibility we examined was to introduce a further sterile right-handed neutrino ν_x. We came to the conclusion that the introduction of such a particle is especially interesting if we introduce a new symmetry that distinguishes it from the original right-handed neutrino ν_R.

In this chapter we will extend the SM gauge group in the simplest possible way, i.e. by introducing an additional $U(1)$ symmetry which will be called the Hidden Sector symmetry as it is supposed to be carried only by particles not belonging to the SM and thus almost completely decouples those particles from the SM particles. Exceptions that do couple the SM sector and the Hidden Sector will be essential in this work as they are necessary to have an effect on the active neutrino masses. The Hidden Sector symmetry

will be denoted by $U(1)_H$.
Adding ν_x requires the mass matrix to be extended. The guiding line of this chapter will be the 3×3 matrix

$$\mathcal{M} = \begin{pmatrix} M_L & m_D & 0 \\ m_D & M_R & M_{Rx} \\ 0 & M_{Rx} & M_x \end{pmatrix} . \tag{5.37}$$

The according basis is given by $(\nu_L, \nu_R^c, \nu_x^c)$ such that we obtain the neutrino mass Lagrangian

$$-\mathscr{L}_m = \frac{1}{2}\overline{n_L}\mathcal{M}n_L^c + h.c.. \tag{5.38}$$

In this way the diagonal entries give the corresponding Majorana masses, while like before m_D denotes the Dirac mass and M_{Rx} gives the coupling between both sterile neutrinos. The element in the upper right and lower left corner describes the mixing between left-handed neutrino and ν_x. We already set it equal to zero as ν_x is supposed to have Hidden Sector (HS) charge while the Higgs doublet and the left-handed doublet does not. Thus there is no way to couple ν_x to the left-handed neutrinos as we do not intend to introduce a scalar $SU(2)$ doublet equipped with a HS charge.
Like in the chapter before we sometimes consider the picture that integrates out ν_x to yield an effective mass M_R.
In this chapter we are going to investigate how the Majorana entries M_R and M_x can be influenced and how this naturally leads to well-known mass matrix structures that describe e.g. the double seesaw mechanism and the inverted seesaw mechanism in a conformally invariant way.
Before studying how to affect the Majorana entries of the mass matrix we want to consider the case of the right-handed neutrino ν_R having a HS charge. In such a theory the term

$$g\overline{L}\tilde{H}\nu_R \tag{5.39}$$

would be forbidden by HS symmetry and no Dirac masses are possible. Therefore the most general mass matrix would look like

$$\mathcal{M} = \begin{pmatrix} M_L & 0 & 0 \\ 0 & M_R & M_{Rx} \\ 0 & M_{Rx} & M_x \end{pmatrix} . \tag{5.40}$$

This matrix is already block-diagonal and the lower right 2×2 matrix will play no role for the left-handed masses. If furthermore $M_L = 0$ like it is in the SM, there will be no active neutrino masses, neither of the Dirac nor the Majorana type. In the picture of integrated out right-handed neutrinos there is no way to connect the left-handed doublets via ν_R or ν_x in the fermion line. In the following we therefore set the HS charge of ν_R to zero.

5.2.1 Modifying the ν_R Majorana Mass

As we are studying conformally invariant theories and we decided that the ν_R does not carry a HS charge, the mass matrix for the conformal standard model would look like

$$\mathcal{M} = \begin{pmatrix} 0 & m_D & 0 \\ m_D & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} . \tag{5.41}$$

Like, however, argued in the chapter before such a theory is phenomenologically inadequate. The easiest way to generate ν_R Majorana masses besides introducing an appropriate scalar that directly replaces M_R is to integrate out the sterile neutrino ν_x like done at the end of the previous chapter. In this case, however, we construct the theory such that the direct term

$$g\varphi\overline{\nu_R}\nu_R^c \tag{5.42}$$

is forbidden by symmetry.

- Particle content: $\nu_R : (1,0,0)$; $\nu_x : (1,0,1)$; $\varphi_1 : (1,0,1)$; $\varphi_2 : (1,0,2)$,

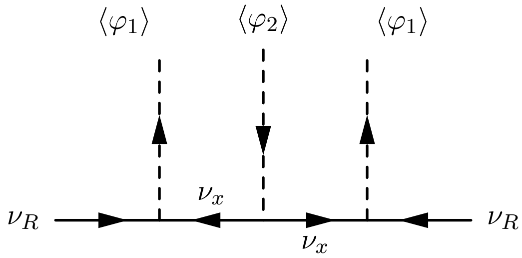
  where the third number in brackets denotes the HS charge. This particle content yields the additional terms

  Yukawa Lagrangian: $-\mathscr{L}_Y = g_1\varphi_1\overline{\nu_R}\nu_x^c + g_2\varphi_2\overline{\nu_x}\nu_x^c + h.c.$

  If φ_1 and φ_2 get a vev this theory yields the mass matrix

$$\mathcal{M} = \begin{pmatrix} 0 & m_D & 0 \\ m_D & 0 & M_{Rx} \\ 0 & M_{Rx} & M_x \end{pmatrix} . \tag{5.43}$$

  This mass matrix represents the double seesaw mechanism. The HS charges of the SM singlet scalars were chosen such that the ν_R Majorana mass cannot be generated directly. We can now integrate out ν_x in the following way:



Having done this we obtain an effective mass M_R and find the contracted mass matrix

$$\mathcal{M} = \begin{pmatrix} 0 & m_D \\ m_D & M_R \end{pmatrix} , \tag{5.44}$$

where M_R can be calculated from the diagram and is given by

$$M_R = \frac{g_1^2}{g_2} \frac{\langle\varphi_1\rangle^2}{\langle\varphi_2\rangle} . \tag{5.45}$$

With φ_1 and φ_2 being approximately of the electroweak scale, M_R will roughly be of electroweak scale as well. This case has been phenomenologically analysed in section 4.5. However, via choosing the coupling constants accordingly, M_R can be tuned to any scale, thus yielding e.g. the Pseudo-Dirac scenario or multi TeV neutrinos, i.e. sterile neutrinos with a mass larger than 1 TeV.

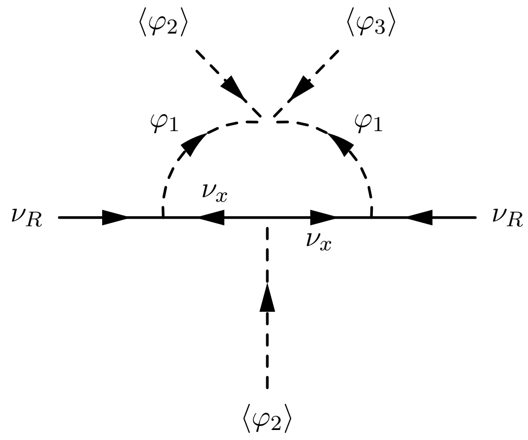
A phenomenologically different scenario occurs if we forbid the vev $\langle\varphi_1\rangle$. Consider the following theory.

- Particle content: $\nu_R : (1,0,0)$; $\nu_x : (1,0,1)$; $\varphi_1 : (1,0,1)$; $\varphi_2 : (1,0,2)$; $\varphi_3 : (1,0,-4)$,

  Note that the newly introduced SM singlet scalar φ_3 does not change the Yukawa Lagrangian. There is, however, an additional potential term.

  Potential: $V = \lambda\varphi_1^2\varphi_2\varphi_3 + h.c.$

  Thus if we forbid, as mentioned, the vev of φ_1, the diagram with the main contribution to M_R is given by



As φ_1 does not get a vev, the original uncontracted mass matrix looks like

$$\mathcal{M} = \begin{pmatrix} 0 & m_D & 0 \\ m_D & 0 & 0 \\ 0 & 0 & M_x \end{pmatrix} \qquad (5.46)$$

After integrating out ν_x like described in the diagram above, the right-handed neutrino ν_R gains a Majorana mass term and the 3×3 mass matrix is contracted to the form

$$\mathcal{M} = \begin{pmatrix} 0 & m_D \\ m_D & M_R \end{pmatrix} . \qquad (5.47)$$

To approximate the scale of M_R we use the following assumptions. A loop built out of particles with masses of electroweak scale, which only involves coupling constants of order 1, will contribute a factor of $1/(16\pi^2) \approx 10^{-2}$, whereas every particle propagator whose scale is by a factor k smaller or bigger than the electroweak scale will contribute a factor $1/k$ if it is a fermion and $1/k^2$ if it is a scalar. Equivalently external lines with a vev contribute a factor k with respect to the vev's relation to electroweak scale. The overall factor for each diagram is the electroweak scale (EWS) itself. Therefore the mass M_R implied by the diagram above is approximately given by

$$M_R \sim 10^{-2} \cdot \frac{\lambda g_1^2}{g_2} \frac{k_2 k_3}{k_1^4 k_2} \cdot EWS = 10^{-2} \cdot \frac{\lambda g_1^2}{g_2} \frac{k_3}{k_1^4} \cdot EWS \,, \qquad (5.48)$$

where k_i is the factor described above for the corresponding vev or propagator contribution of φ_i. If we now assume that $\langle\varphi_3\rangle$ is of electroweak scale, i.e. $k_3 = 1$ and the mass scale of φ_1 is one order of magnitude above EWS, i.e. $k_1 = 10$ and all couplings are of the order 1, we obtain

$$M_R \sim 10^{-6} \cdot EWS\,. \tag{5.49}$$

In our phenomenological analysis of sterile neutrinos we saw that this lies within the non-viable region in parameter space. The aim of this theory, however, is to produce Pseudo-Dirac neutrinos. We saw that the viable region for Pseudo-Dirac neutrinos begins at approximately $10^{-14} \cdot EWS$. Before we assumed that all couplings are proportional to 1. Tuning those can help further suppressing the scale. Consider the following reasonable assumptions:
Like before k_3 is assumed to be 1, whereas now we take $k_1 = 10^{1.5}$ which is still realistic. Furthermore we require all couplings to be of the same scale, but this time $\sim 10^{-3}$. In this case we reach $M_R \sim 10^{-14} \cdot EWS$. We see that it is indeed possible within this theory to generate Pseudo-Dirac neutrinos. Coupling constants smaller than one are the reason for perturbation theory to work and another reason why loop diagrams are suppressed with respect to tree-diagrams. It is therefore understandable that by decreasing the coupling constants the mass M_R can be further suppressed in this theory.

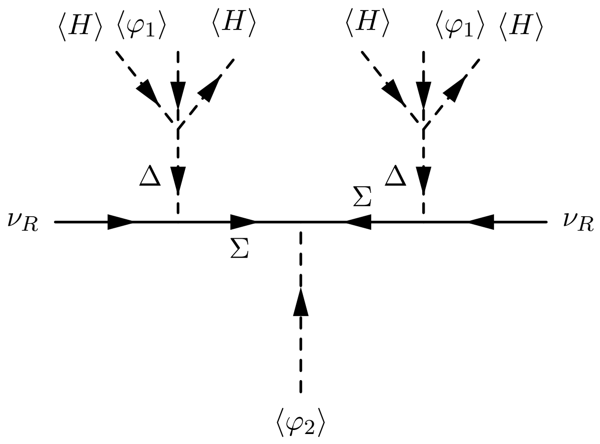
We succeeded in modifying the right-handed neutrino mass by integrating out the other sterile neutrino. In that sense we did not really simply modify the ν_R Majorana entry of the 3×3 matrix but rather contracted it down to a 2×2 matrix. In the following we want to find theories that actually maintain the original shape of the mass matrix but still generate the ν_R mass. The next theory we want to consider achieves this by introducing a scalar and a fermion triplet.

- Particle content: $\nu_R : (1,0,0);\ \nu_x : (1,0,1);\ \Sigma : (3,0,1);\ \Delta : (3,0,1);\ H : (2,1,0);$
  $\varphi_1 : (1,0,1);\ \varphi_2 : (1,0,2)$

  Yukawa Lagrangian: $-\mathscr{L}_Y = g_\Delta \,\mathrm{Tr}\,[\overline{\Sigma}\Delta\nu_R] + g_\Sigma \,\mathrm{Tr}\,[\varphi_2\overline{\Sigma^c}\Sigma] + h.c.$

  Potential: $V = \lambda\varphi_1 H^T i\sigma_2 \mathbf{\Delta}^\dagger \tilde{H} + h.c.$

  Note that we only displayed terms in the Yukawa Lagrangian and the potential that are relevant for the lowest order diagram of right-handed neutrino mass generation. This diagram is given by



This is basically the same theory as in section 5.1.5. The difference, however, is that this is the only tree-level diagram if Δ does not get a vev as the term $\varphi\overline{\nu_R}\nu_R^c$ does not exist. The right-handed mass M_R is therefore to first order given by

$$M_R = \frac{\lambda^2 g_\Delta^2}{g_\Sigma}\left(\frac{\langle H\rangle}{M_\Delta}\right)^4 \frac{\langle\varphi_1\rangle^2}{\langle\varphi_2\rangle}\,. \tag{5.50}$$

Unlike in section 5.1.5 the factor $\left(\frac{\langle H\rangle}{M_\Delta}\right)^4$ can now be the reason for the suppression of the right-handed neutrino mass. The aim as before is to generate Pseudo-Dirac neutrinos. If we choose the mass of Δ to be one order of magnitude below the electroweak scale, all occurring couplings of the order 10^{-3} and $\langle\varphi_1\rangle$ one order of magnitude below EWS and $\langle\varphi_2\rangle$ of electroweak scale, we obtain for the right-handed mass

$$M_R \sim 10^{-15}\cdot EWS\,. \tag{5.51}$$

This lies in the range of Pseudo-Dirac neutrinos.
We have to note that M_Δ itself is not independent of the vacuum expectation values and the couplings. The propagator displayed in the diagram above gets its mass through potential couplings given by

$$\begin{aligned} V = \lambda_{\Delta H}(\Delta^\dagger\Delta)(H^\dagger H) + \lambda_{\Delta\varphi_1}(\Delta^\dagger\Delta)(\varphi_1^\dagger\varphi_1) \\ +\lambda_{\Delta\varphi_2}(\Delta^\dagger\Delta)(\varphi_2^\dagger\varphi_2) \end{aligned}\,. \tag{5.52}$$

To first order M_Δ is therefore given by

$$M_\Delta = 2\lambda_{\Delta H}\langle H\rangle^2 + 2\lambda_{\Delta\varphi_1}\langle\varphi_1\rangle^2 + 2\lambda_{\Delta\varphi_2}\langle\varphi_2\rangle^2\,. \tag{5.53}$$

If we assumed that all couplings are of order 10^{-3} and the vevs are like given above the mass would be well below EWS. Therefore some degree of fine-tuning is necessary in the sense that potential coupling constants have to differ by three orders of magnitude with respect to Yukawa couplings. We can only achieve the scale of M_Δ to be bigger than that of $\langle H \rangle$ if at least one potential coupling constant is of the order 1 and the corresponding vev is bigger than that of $\langle H \rangle$. Ideally in our example we take all potential coupling constants of the order 1, the vev of φ_2 one order above electroweak scale and the rest like before. We took all potential coupling constants of the order 1 to avoid another fine-tuning. In this case we obtain

$$M_R \sim \left(10^{-13} - 10^{-14}\right) \cdot EWS \,. \tag{5.54}$$

Although we did not write it down explicitly in the Yukawa Lagrangian the theory contains the mixing term $\varphi_1 \overline{\nu_R} \nu_x^c$ and the term $\varphi_2 \overline{\nu_x} \nu_x^c$. This means that the mass matrix is given by

$$\mathcal{M} = \begin{pmatrix} 0 & m_D & 0 \\ m_D & M_R & M_{Rx} \\ 0 & M_{Rx} & M_x \end{pmatrix} . \tag{5.55}$$

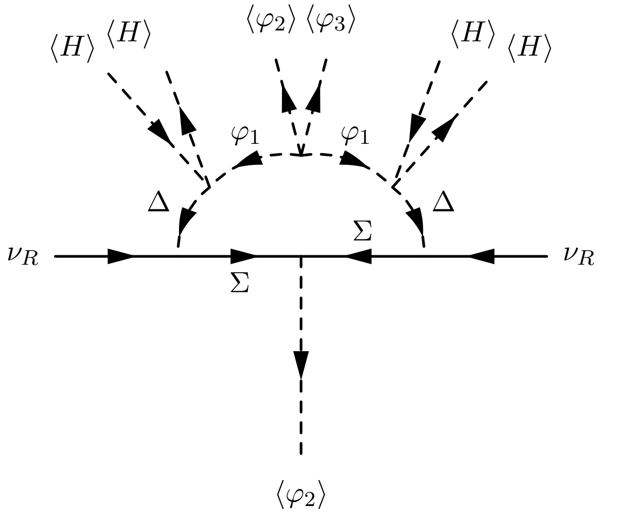
As seen before radiative generation of masses has the feature of naturally being suppressed by a common factor of $1/(16\pi^2)$ per loop and a bigger number of couplings constants being involved. We will therefore modify the previous theory such that the lowest order contributing diagram is a one-loop diagram. First of all we forbid the vev of φ_1.

- Particle content: $\nu_R : (1,0,0);\ \Sigma : (3,0,1);\ \Delta : (3,0,1);\ H : (2,1,0);$
  $\varphi_1 : (1,0,1);\ \varphi_2 : (1,0,2);\ \varphi_3 : (1,0,-4)$

  The Yukawa Lagrangian is the same as in the previous theory, while we get an additional potential term.

  Potential: $V = \lambda \varphi_1 H^T i\sigma_2 \mathbf{\Delta}^\dagger \tilde{H} + \lambda' \varphi_1^2 \varphi_2 \varphi_3 + h.c. + ...$

  Forbidding $\langle \varphi_1 \rangle$ like mentioned the lowest order diagram contributing to the right-handed neutrino mass is given by



Taking the approximations for the evaluation of loop diagrams described before, we find the approximate mass

$$M_R \sim 10^{-2} \cdot \frac{g_\Delta^2 \lambda^2 \lambda'}{g_\Sigma} \frac{k_3}{k_1^4 k_\Delta^4} \cdot EWS \tag{5.56}$$

Assuming that all potential coupling constants are of order 1, $\langle\varphi_1\rangle$ is one order above EWS and $\langle\varphi_3\rangle$ is of electroweak scale, we find

$$M_R \sim \frac{g_\Delta^2}{g_\Sigma} \cdot 10^{-10} \cdot EWS\,, \tag{5.57}$$

where we have used that k_1 and k_Δ have to be of the same order because all potential coupling constants have been chosen to be of the order 1 and thus the mass of Δ is mainly determined by $\langle\varphi_1\rangle$ which is one order of magnitude above EWS. Yukawa couplings of the order 10^{-4} or smaller yield again Pseudo-Dirac neutrinos. Note furthermore that the factors to the power 4 in the denominator are very sensitive to changes and can make M_R very quickly even smaller.
As a consequence of φ_1 not getting a vev, there is no term $\varphi_1 \overline{\nu_R} \nu_x^c$ and thus the mass matrix is given by

$$\mathcal{M} = \begin{pmatrix} 0 & m_D & 0 \\ m_D & M_R & 0 \\ 0 & 0 & M_x \end{pmatrix} . \tag{5.58}$$

In this case, however, this does not mean that there is no connection between SM sector and Hidden Sector as φ_1 can form a loop and connect via a potential coupling to vevs of different scalars just like in the second theory of this section.

5.2.2 Modifying the ν_x Majorana Mass

The last task is see how other theoretical extensions influence the ν_x Majorana mass. We want to do this in a way such that the right-handed neutrino mass of ν_R stays zero such that we get the following mass matrix structure

$$\mathcal{M} = \begin{pmatrix} 0 & m_D & 0 \\ m_D & 0 & M_{Rx} \\ 0 & M_{Rx} & M_x \end{pmatrix} . \tag{5.59}$$

We already generated this structure before where the Majorana mass of ν_x was generated via the simple Yukawa term

$$g\varphi\overline{\nu_x}\nu_x^c . \tag{5.60}$$

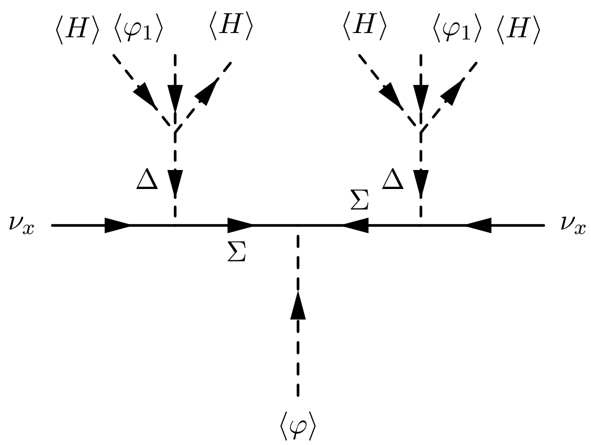
There a Majorana mass term is created by φ getting a vev. Our aim in this section, however, is to forbid this kind of term and generate naturally small ν_x Majorana masses. In this case the mass matrix in eq. (5.59) yields an inverse seesaw mechanism. The theories we consider here are very similar to those of the previous section.

- Particle content: $\nu_x : (1,0,1)$; $\Sigma : (3,0,-2)$; $H : (2,1,0)$; $\varphi_1 : (1,0,-3)$; $\varphi_2 : (1,0,-4)$; $\Delta : (3,0,-3)$

  Yukawa Lagrangian: $-\mathscr{L}_Y = g_\Delta \operatorname{Tr}[\overline{\Sigma}\Delta\nu_x] + g_\Sigma \operatorname{Tr}[\varphi_2\overline{\Sigma^c}\Sigma] + h.c.$

  Potential: $V = \lambda\varphi_1 H^T i\sigma_2 \mathbf{\Delta}^\dagger \tilde{H} + h.c. + ...$

  From this we can build the following diagram



Note that as required the coupling $\varphi\overline{\nu_x}\nu_x^c$ is forbidden and there is no direct or effective right-handed neutrino mass for ν_R possible as there are no appropriate Yukawa couplings. Note furthermore, so far the term $\varphi\overline{\nu_R}\nu_x^c$ is not possible as well and thus the mass matrix until now reads

$$\mathcal{M} = \begin{pmatrix} 0 & m_D & 0 \\ m_D & 0 & 0 \\ 0 & 0 & M_x \end{pmatrix} . \tag{5.61}$$

There is no connection between Hidden Sector and SM sector. Thus there is no way to influence the active neutrino mass via the Hidden Sector. This is not what we are aiming for. We therefore have to introduce another scalar φ_4 with HS charge 1 such that we obtain the connecting Yukawa term $g_{Rx}\varphi_4\overline{\nu_R}\nu_x^c$ and the mass matrix is actually given by eq. (5.59).

Like seen in the section before the Majorana mass of ν_x is given by

$$M_x = \frac{\lambda^2 g_\Delta^2}{g_\Sigma}\left(\frac{\langle H\rangle}{M_\Delta}\right)^4 \frac{\langle\varphi_1\rangle^2}{\langle\varphi_2\rangle} ,$$

where the parameters can be tuned such that the mass becomes very small and the requirements for the inverse seesaw mechanism are indeed fulfilled. On the other hand we can tune parameters such that M_x becomes very big. This in turn yields that if φ_4 gets a vev, via a seesaw mechansim the mass M_R is very much suppressed and we find again the scenario of Pseudo-Dirac neutrinos.

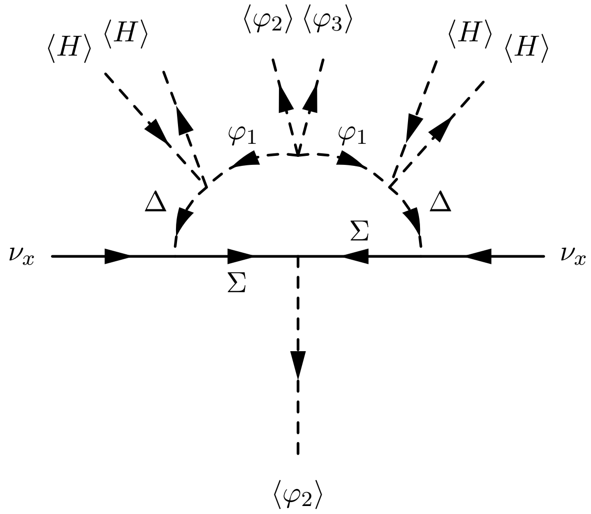
Like before we want to try to find a theory that generates the ν_x Majorana masses radiatively. We do this by forbidding the vev of φ_1 and introducing a further scalar φ_3 and adjusting the HS charges.

- Particle content: $\nu_x : (1,0,1);\ \Sigma : (3,0,-2);\ H : (2,1,0);\ \varphi_1 : (1,0,-3);$
  $\varphi_2 : (1,0,-4);\ \varphi_3 : (1,0,10);\ \Delta : (3,0,-3)$

  Note that with this choice of HS charges we meet the requirements of zero right-handed Majorana mass and non-direct generation of the ν_x mass. The Yukawa Lagrangian is the same as in the theory before. The potential, however, is extended.

  Potential: $V = \lambda\varphi_1^2\varphi_2\varphi_3 + h.c.$

  We obtain the following diagram



As before the Majorana mass of ν_x can be approximated by

$$M_R \sim 10^{-2} \cdot \frac{g_\Delta^2 \lambda^2 \lambda'}{g_\Sigma} \frac{k_3}{k_1^4 k_\Delta^4} \cdot EWS\,. \tag{5.62}$$

We see that in this setup the ν_x mass is naturally small when the couplings are of order 1. Note that as before we need another scalar φ_4 for the connection between SM sector and Hidden Sector.

5.3 Summary of the Models and Phenomenological Discussion

Tables 5.2 and 5.3 summarize the conformally invariant models that have been discussed in this chapter. The first table refers to models within the SM gauge group and displays the corresponding particle content and the non-conformal motivations. Furthermore, it is indicated which models yield neutrino masses and if they can imply the right Higgs mass. Beyond that the most important phenomenological properties are mentioned. Models modifying the left-handed Majorana masses are presented in table 5.2 and models modifying the right-handed Majorana masses are presented in table 5.3.
The second table refers to conformal models with an additional $U(1)$ symmetry. For these theories the particle content with their $U(1)$ charge was displayed. In addition the vev structure of the different models has been highlighted. Like before the most important phenomenological properties are mentioned.

Phenomenological Discussion

In order to study the experimental verifiability of the discussed models we will arrange them in four phenomenological classes. These are given by

- the sub TeV seesaw scenario
- the Pseudo-Dirac scenario
- the case of pure left-handed Majorana neutrinos
- the multi TeV seesaw scenario.

The sub TeV seesaw scenario refers to the regions in parameter space of Dirac and Yukawa couplings given in figures 4.3, 4.4 and 4.5, while the Pseudo-Dirac scenario refers to the region given in fig. 4.8 in the same parameter space.
The case of pure left-handed Majorana neutrinos would be at the origin of this parameter space. It implies that no right-handed neutrinos are involved.
The multi TeV seesaw scenario lies in the continuation of the viable regions in figures 4.3, 4.4 and 4.5 to higher Yukawa couplings with Majorana masses bigger than 1 TeV.
Of course the assignment of theories to the different groups is not always unique and depends on the actual choice of the coupling constants. The assignments of the different theories are summarized in tables 5.2 and 5.3.
We will see which observations can dismiss or favour a certain phenomenological class. This analysis is based on the occurrence of the neutrinoless double beta decay ($0\nu\beta\beta$) and of lepton flavour violating (LFV) processes of charged leptons and on the non-unitarity of the PMNS matrix.
We saw that for sub TeV neutrinos all three effects can play a role. In our parameter scan in section 4.5 we found out that at the moment the effective mass of the electron neutrino is the most limiting one. Improving the constraints on the effective mass might push this line over the perturbative boundary and the line of smallest possible active-sterile mixing given by eq. (4.51) and would thus exclude sub TeV scenarios. However, the upper bound for the effective electron neutrino mass would have to be placed below 10^{-10} eV which will not be experimentally realized within the near future. Furthermore, the smallest values for the non-unitarity in figures 4.3, 4.4 and 4.5 are around 10^{-10} to 10^{-12}. Therefore, if experiments set an upper limit for non-unitarity below this threshold, this scenario could be excluded as well. Accordingly, better experimental bounds on LFV processes can achieve the same. Thus the Meissner-Nicolai model which is part of this group could be mostly excluded. However, we found that it still yields a further viable region which represents the Pseudo-Dirac scenario.

This region might finally be excluded if we found the neutrinoless double beta decay with an effective electron neutrino mass that is roughly bigger than 10^{-2} eV. On the other hand a paper by P.C. Holanda and A.Yu. Smirnov explains how an additional very light sterile neutrino can influence the energy spectra of the solar neutrino events [53]. They argue that this can explain recent results from the SNO, Super-Kamiokande and Borexino experiments which do not show the expected upturn of events at low energies. In the same way the Pseudo-Dirac scenario which includes 3 additional light neutrino states can be tested. The general effect of additional light neutrino states close to the active states is the oscillation into these light states, which results in small wiggles around the basic probability oscillation curve.

The third group refers to the case of pure left-handed Majorana neutrinos. This group is characterized by the fact that it does not yield LFV processes and non-unitarity effects. This means that if any LFV process or a non-unitarity effect could be measured with high accuracy this scenario could be excluded. The conformal form of the type II seesaw mechanism and of the Zee-Babu model would thus be excluded.

The fourth group is the multi TeV seesaw scenario. Large parts of this region, however, can be excluded as they are not consistent with electroweak observables. This is the case for the region in parameter space with a high degree of non-unitarity.

A certain realization of this multi TeV scenario is the inverse seesaw mechanism that has been realized by theory 5 and 6 of table 5.3. It has to be noted that this scenario naturally yields large non-unitarity η which is given by ([54])

$$\eta = \frac{1}{2} m_D^\dagger \left(M_{Rx}^{-1}\right)^* \left(M_{Rx}^{-1}\right)^T m_D \,. \tag{5.63}$$

This means that if the scale of the mixing M_{Rx} is one order of magnitude above electroweak scale the mixing is of order 10^{-2} which is at the boundary of allowed non-unitarity.

Although it is experimentally almost impossible to exclude certain regions completely, improvements in ongoing and future experiments will more and more shape the regions of viable phenomenology. After all it is still possible that non-unitarity effects or the Majorana character of neutrinos can be detected. In this case a large class of theories will have to be dismissed. In so far neutrino physics is on the borderline of detecting new physics.

Beyond testing the properties of the different groups the theories themselves can be tested by looking for corresponding additional particles. In a conformally invariant framework like presented in this work, this becomes especially fruitful as the mass scales of the particles have to lie in the TeV range. This is because conformal invariance forbids the introduction of arbitrary scales. Instead these have to be generated by the vevs of scalars. As this vevs in turn are generated radiatively it is reasonable to assume that all vevs are approximately of the same scale, i.e. that there are no unnatural hierarchies. Hence, all vevs should be approximately of TeV scale. Current collider experiments are on the borderline of testing these energy ranges and could thus be able to detect particles of this scale.

For each phenomenological class there are some theories that are very promising and should be further studied regarding the viability of radiative symmetry breaking and the phenomenological properties. E.g. the easiest model realizing the sub TeV conditions is the Meissner-Nicolai model. Theory 2 of table 5.3 yields conditions for the Pseudo-Dirac scenario in a very natural way by loop suppression. It has to be investigated if the suggested vev structure can indeed be realized by radiative symmetry breaking. A very interesting model for the case of pure left-handed Majorana neutrinos is the conformally invariant Zee-Babu model as it suppresses the neutrino masses by two loops. Finally the first theory modifying the right-handed Majorana masses in table 5.2 is the simplest way to generate multi TeV neutrinos. It has to be checked if the suggested scales can be realized by radiative symmetry breaking.

Conformal Mass Models within the SM Gauge Group

particle content	non-conformal motivation	neutrino masses	correct Higgs mass	phenomenological note
Left-Handed Majorana Masses				
Conformal SM (CSM)	/	**No**	**No**	This theory does not yield neutrino masses.
CSM + $\nu_R : (1,0)$	Seesaw type I	**Yes**	**No**	Neutrinos in this theory are of Dirac type.
CSM + $\nu_R : (1,0)$ + $\varphi : (1,0)$	Seesaw type I	**Yes**	**Yes**	In dependence of the coupling constants this theory can yield Sub TeV or Pseudo-Dirac neutrinos.
CSM + $\Delta : (3,-2)$	Seesaw type II	**Yes**	**No**	This theory yields pure left-handed Majorana neutrinos.
CSM + $\Delta : (3,-2)$ + $\varphi : (1,0)$	Seesaw type II	**Yes**	**Yes**	This theory yields pure left-handed Majorana neutrinos as well.
CSM + $\nu_R : (1,0)$ + $\varphi : (1,0)$ + $\Delta : (3,-2)$	Seesaw type I/II	**Yes**	**Yes**	Sub TeV and Pseudo-Dirac neutrinos are possible.
CSM + $\delta_- : (1,-2)$	/	**No**	**No**	Neutrinos remain massless.
CSM + $\delta_- : (1,-2)$ + $\Delta : (3,-2)$	/	**Yes**	**No**	The additional δ_- only contributes corrections to the masses.
CSM + $\Sigma : (3,0)$	Seesaw type III	**No**	**No**	Neutrinos remain massless.
CSM + $\Sigma : (3,0)$ + $\varphi : (1,0)$	Seesaw type III	**Yes**	**Yes**	This theory yields the same neutrino phenomenology like the conformal Seesaw type I.
CSM + $\delta_- : (1,-2)$ + $\epsilon_{++} : (1,4)$ + $\varphi : (1,0)$	Zee-Babu	**Yes**	**Yes**	Pure left-handed Majorana neutrino masses suppressed by 2 loops.
Right-Handed Majorana Masses				
CSM + $\nu_R : (1,0)$ + $\Sigma : (3,0)$ + $\Delta : (3,0)$ + $\varphi : (1,0)$	/	**Yes**	**Yes**	This theory can generate conditions for the Pseudo-Dirac, the Sub TeV but also for the multi TeV scenario.
CSM + $\nu_R : (1,0)$ + $\nu_x : (1,0)$ + $\varphi : (1,0)$	/	**Yes**	**Yes**	The extension by further sterile neutrinos is trivial if they cannot be distinguished from the original sterile neutrinos.

TABLE 5.2: Summary of different conformally invariant models for the generation of neutrino masses within the SM gauge group.

Conformal Mass Models with Additional U(1) Symmetry

#	particle content	$U(1)_H$	vev structure	phenomenological note

ν_R Majorana Masses

#	particle content	$U(1)_H$	vev structure	phenomenological note
1	$\nu_R : (1,0)$	0	all scalars get a vev	The double seesaw mass structure is implied.
	$\nu_x : (1,0)$	1		Pseudo-Dirac, sub TeV and
	$\varphi_1 : (1,0)$	1		multi TeV scenarios
	$\varphi_2 : (1,0)$	2		are possible.
2	theory 1 +	theory 1	φ_1 gets no vev	radiative model,
	$\varphi_3 : (1,0)$	-4		implies Pseudo-Dirac scenario
3	$\nu_R : (1,0)$	0	all scalars get a vev	
	$\Sigma : (3,0)$	1		Pseudo-Dirac, sub TeV and
	$\Delta : (3,0)$	1		multi TeV scenarios
	$\varphi_1 : (1,0)$	1		are possible.
	$\varphi_2 : (1,0)$	2		
4	theory 3 +	theory 3	φ_1 gets no vev	radiative model,
	$\varphi_3 : (1,0)$	-4		implies Pseudo-Dirac scenario

ν_x Majorana Masses

#	particle content	$U(1)_H$	vev structure	phenomenological note
5	$\nu_R : (1,0)$	0	all scalars get a vev	generates small ν_x mass,
	$\nu_x : (1,0)$	1		implies the inverse seesaw scenario
	$\Sigma : (3,0)$	-2		
	$\Delta : (3,0)$	-3		
	$\varphi_1 : (1,0)$	-3		
	$\varphi_2 : (1,0)$	-4		
6	theory 5 +	theory 5	φ_1 gets no vev	radiative model,
	$\varphi_3 : (1,0)$	10		implies the inverse seesaw scenario

TABLE 5.3: Summary of different conformally invariant models for the generation of neutrino masses with an additional HS symmetry.

Chapter 6

Conclusion

With a huge number of ongoing experiments around the world and many theorists being involved neutrino physics may be one of the most active research fields in physics. The reason for its popularity might be the fact that it is very promising to shed light on fundamental principles of nature that lie beyond the predictions of the Standard Model. In the upcoming years, thanks to a vast experimental commitment, we can be optimistic to learn more about the nature of neutrinos which concerns questions like: "Are neutrinos of Dirac or Majorana type?", and: "Which is the true secret behind the smallness of neutrino masses?" Better and better experimental results will give theorists a clue which of their ideas has to be given up and which might lead to a more profound understanding of nature. Thus, neutrino physics may not only be a research field for itself but might deliver insight into other unsolved questions in physics.

In this way the issue of conformal invariance as the solution to the Hierarchy Problem could be considered from a different perspective. If we found a conformally invariant theory that generated neutrino masses with their required smallness in a very natural way and that led rather automatically to the precisely known SM phenomenology, further yielding a very characteristic Beyond the Standard Model phenomenology that can be tested in future experiments, then the discussions about conformal invariance would take a completely new direction, as after all nature would be described by a conformally invariant theory.

In this sense several conformally invariant models were presented that can lead to small neutrino masses. We began by reviewing the principles of scale and conformal transformations and the argument of conformally invariant theories solving the Hierarchy Problem. We continued by showing that conformally invariant theories can indeed yield massive particles and introduced the effective potential for this purpose.

After revisiting general subjects of neutrino physics, we introduced the Meissner-Nicolai

model in chapter 4, which extended the SM particle content by three right-handed neutrinos and one scalar singlet. In this context we performed a parameter scan of the involved Dirac and Majorana coupling constants yielding three results. First of all it was generally possible with the vev of the singlet scalar being of electroweak scale to choose the coupling constants smaller than 1 such that neutrino masses could be generated in a phenomenologically acceptable way. Secondly we could even within the regions of phenomenological viability approach Majorana and Dirac coupling constants to a ratio of approximately 10^{-2} with respect to each other. Last we also saw that there are basically 2 viable regions in parameter space. The first one corresponds to sub TeV neutrinos, the other to Pseudo-Dirac neutrinos.
In the first section of chapter 5 we initially investigated several models to modify the left-handed Majorana entries of the one-flavour 2×2 mass matrix. We saw that terms which were possible in the non-conformal case could not be introduced in the conformal case. Thus mass terms possible in non-conformal theories were only there if we adjusted the particle content appropriately. This shows a feature of non-conformal theories that the origin and the scale of neutrino masses can be influenced very strongly by the introduction of a singlet scalar replacing the non-dynamical masses.
After we got to know the general way how to build neutrino masses in conformally invariant theories, we used general conformal building rules to deduce two topological lemmata, one concerning the vev structure of conformal diagrams and one concerning the possibility or impossibility of fully radiative diagrams.
Afterwards we attended the modification of the right-handed Majorana masses. We discussed two ways to do that. One of them introduces a triplet fermion and it was shown that with natural assumptions the scale of M_R can easily be pushed 4 orders of magnitude above electroweak scale and thus a natural origin of the seesaw mechanism is implied.
In the second section of chapter 5 an additional U(1) symmetry was introduced, hence creating a Hidden Sector separated from the SM. A further sterile right-handed neutrino with HS charge 1 was introduced extending the mass matrix to a 3×3 shape. With this additional symmetry and the right choice of the particle content and the vev structure we were able to naturally generate mass structures that yielded the double seesaw mechanism and the inverse seesaw mechanism. Partly this was achieved by choosing the vev structure and the particle content such that neutrino masses were generated radiatively.

To conclude, conformal symmetry forbids a large number of neutrino mass models. It is not allowed to arbitrarily introduce new mass scales. All masses in the theory are determined by the running of the couplings and boundary conditions, e.g. at the Planck scale. Accepting conformal symmetry as a fundamental law of nature changes the way how we find neutrino mass models. Instead of adapting the mass parameters arbitrarily and thus inducing seesaw conditions we have to choose the particle content and the couplings in a way that naturally implies such scales. Thus, in a way, conformal symmetric theories are more compelling. Assuming that from phenomenological observations one neutrino mass mechanism suggests itself, a certain particle content will be implied which would be very characteristic for a theory. I.e. in conformally invariant theories phenomenological constrictions call for a dynamical explanation rather than a mere introduction of new scales.

The other way round, the identification of a certain neutrino mass mechanism as the truth and the detection of certain particles could hint at a conformally invariant theory as the underlying theory of nature.

Beyond that, replacing masses of a non-conformal theory by dynamical fields yields two further features. On the one hand dynamical fields can carry charges. Thus, by introducing an additional symmetry, certain coupling and mass terms can be argued to be existent or non-existent based on symmetry arguments. On the other hand the vev structure can be used to forbid or allow mass terms and to influence the topologies of diagrams.

This work presented a collection of conformally invariant theories that generate neutrino masses. It has, however, not been shown how the effective potential of the different theories would look like, if these potentials would indeed yield radiative symmetry breaking and if the suggested vev structure can be realized such that the proper Higgs mass is implied. Furthermore, in future works it could be investigated how different Planck scale boundary conditions influence the vev structure.

On the other hand detailed phenomenological implications of the different theories have to be elaborated in order to test if they have indeed the potential to describe nature properly.

Conformally invariant neutrino mass models were investigated within the SM gauge group and with an additional Hidden Sector which exists almost independent of the SM. We did not study neutrino mass models in conformally invariant theories within gauge group extension that embed the SM. E.g. to find conformal neutrino mass models within the left-right symmetric model, which extends the SM gauge group by an additional $SU(2)_R$, would be an interesting task for future works.

Appendix A

Renormalization Group Improved Effective Potential

This chapter deals with the renormalization group improved effective potential and its necessity in scalar ϕ^4-theory concerning radiative symmetry breaking. In chapter 2 we saw that the effective potential for ϕ^4-theory was given by eq. (2.71). Thus differentiating eq. (2.71) once with respect to ϕ_0 and setting it equal to 0 yields the extrema $\phi_0 = 0$ and $\lambda ln\frac{\phi_0}{M} \sim -\frac{16}{3}\pi^2$, where the extremum at the origin, however, is now a maximum, while the other one really is a minimum. Naively we could say that the symmetry is indeed radiatively broken. But we should be aware that this result is a one-loop approximation. To do things properly we should first have a closer look at the structure of higher order loop corrections. It becomes clear that every additional loop also contributes an additional factor $[\ln\left(\frac{\phi_0^2}{M^2}\right)]$ and thus the n-loop contributions to the effective potential have a factor $\lambda^{n+1}[\ln\left(\frac{\phi_0^2}{M^2}\right)]^n$. As we found out, $\lambda\ln\frac{\phi_0}{M} \sim -\frac{16}{3}\pi^2$ at the minimum, and its absolute value is bigger than 1. It is now evident that perturbation theory breaks down if we are interested in field values that are of the order of the symmetry breaking scale. The effective potential to one-loop order or any other order in perturbation theory is thus not reliable and fails to describe spontaneous symmetry breakdown.

A second flaw of the one-loop approximation of the effective potential becomes apparent when analyzing the dependence of the effective potential on the arbitrary parameter M. Due to its arbitrariness the effective potential may not depend on it. Investigating eq. (2.71), however, shows that this requirement is not satisfied. To meet this requirement we have to impose the condition

$$M\frac{dV_{\text{eff}}}{dM} = \left[M\frac{\partial}{\partial M} + \beta\frac{\partial}{\partial\lambda} - \gamma\phi\frac{\partial}{\partial\phi}\right]V_{\text{eff}} = 0, \qquad \text{(A.1)}$$

which is called the Renormalization Group Equation (RGE), where

$$\beta = M\frac{d\lambda}{dM} \tag{A.2}$$

and

$$\gamma\frac{\phi}{M} = -\frac{d\phi}{dM} \tag{A.3}$$

are called the beta function and the anomalous dimension of the wave function, respectively. If we knew both exactly, the solution of the RGE would be the exact effective potential. However, we do not know them exactly, but to a certain order in perturbation theory. Unlike the effective potential, which is really a power series in $\lambda \ln \frac{\phi_0}{M}$, the beta function and the anomalous dimension are simply expansions in the coupling. As the coupling can be guaranteed to be small, perturabation theory is indeed valid. In so far the effective potential obtained by the RGE would really be an improvement of the former result. By considering the running of the parameters we thus can also get rid of the first flaw, the breakdown of perturbation theory. The result of the RGE will be called renormalization group improved effective potential.

In order ro really appreciate the RGE's and to understand the result in eq. (2.71) in the context of the RGE's we will try to evaluate them for the potential $V = \frac{\lambda}{4!}\phi^4$. For dimensional reasons the effective potential has to have the form

$$V_{\text{eff}} = Y(\lambda, t)\phi_0^4, \tag{A.4}$$

where $t = \ln \frac{\phi_0}{M}$. In terms of this the RGE's can be rewritten in the form

$$\left(-\frac{\partial}{\partial t} + \bar{\beta}\frac{\partial}{\partial\lambda} - 4\bar{\gamma}\right) Y(\lambda, t) = 0, \tag{A.5}$$

where

$$\bar{\beta} = \frac{\beta}{1+\gamma} \quad \text{and} \quad \bar{\gamma} = \frac{\gamma}{1+\gamma}.$$

The solution is given by

$$Y(\lambda, t) = f(\lambda'(t,\lambda)) \exp\left(-4\int_0^t \mathrm{d}t'\, \bar{\gamma}(\lambda'(\lambda, t))\right), \tag{A.6}$$

where f is an arbitrary function depending on λ' and $\lambda'(\lambda, t)$ is given by the solution of the equation

$$\frac{d\lambda'}{dt} = \bar{\beta}(\lambda') \tag{A.7}$$

and by the boundary condition $\lambda'(\lambda, 0) = \lambda$. The same renormalization condition fixes the function f and yields

$$Y(\lambda, t) = \frac{1}{4!}\lambda'(\lambda, t) \exp\left(-4\int_0^t \mathrm{d}t'\, \bar{\gamma}(\lambda'(\lambda, t))\right). \tag{A.8}$$

It is now convenient to rederive the effective potential given by eq. (2.71) using this result. Assuming $\beta = const.$ and $\gamma = 0$, we find

$$V_{\text{eff}} = \frac{\phi_0^4}{4!}\left[\beta \ln\frac{\phi_0}{M} + const.\right], \tag{A.9}$$

which structurally is the result from eq. (2.71). We now see that when calculating the effective potential to one-loop order and renormalizing it, we implicitly assume that the beta function of the coupling λ is constant and that the anomalous dimension of the wave function $\gamma = 0$. In the case of small excursions around M in field space, this result is perfectly fine. This claim is supported by the fact that dealing with field values $\phi_0 \sim M$ gives values around 0 for $\ln\frac{\phi_0}{M}$ and perturbation theory works out well, restoring the significance of the one-loop result of eq. (2.71).

We proceed in finding the renormalization group improved effective potential. To do so we first have to find an expression for the anomalous dimension. The dependence of ϕ_0 on M lies completely in the field renormalization factor Z, which is defined in the following way

$$(\partial_\mu \phi_0')^2 = (\partial_\mu \phi_0)^2 Z(\ln(M'/M), \lambda), \tag{A.10}$$

where ϕ_0' is the new renormalized field. From this we find

$$\phi_0' = \phi_0 Z(\ln(M'/M), \lambda)^{1/2}. \tag{A.11}$$

With the definition of the anomalous dimension (A.3) we finally get

$$\gamma(\lambda) = -\frac{d\ln\sqrt{Z}}{dM}. \tag{A.12}$$

Next, from eq. (2.72), we can calculate the beta function, which is defined in eq. (A.2),

$$\beta = \frac{3\lambda^2}{16\pi^2}. \tag{A.13}$$

To one-loop order the anomalous dimension of the wave function is found to be equal to 1. Plugging both into eq. (A.7) yields the differential equation

$$\frac{d\lambda'}{dt} = \frac{9\lambda'^2}{8\pi^2} \tag{A.14}$$

which has the solution

$$\lambda' = \frac{\lambda}{1 - 3\lambda t/16\pi^2}, \tag{A.15}$$

already reflecting the correct renormalization for $t = 0$, which becomes clear when writing down the final result

$$V_{\text{eff}} = \frac{1}{4!}\left(\frac{\lambda}{1 - 3\lambda t/16\pi^2}\right)\phi_0^4. \tag{A.16}$$

This is called the renormalization group improved effective potential. It is valid for all $t = \ln\frac{\phi_0}{M}$ and thus also priviledged to describe the possible breakdown of symmetry. For small values of t it reproduces eq. (2.71). As a matter of fact the former maximum of the one-loop result at the origin has become a minimum and the minimum of the one-loop result has vanished. Consequently there is no radiative symmetry breaking in ϕ^4-theory. Apparently it was an error generated by the inaccurate loop expansion.

Appendix B

The Coleman-Weinberg and the General Effective Potential

In the follwing we want to calculate the Coleman-Weinberg (CW) potential for scalar electrodynamics rather explicitly. To do so first the effective potential is derived for the most general theory containing a collection of scalars and vector bosons. This is enough to derive the explicit form of the CW potential. To conclude we will also introduce a collection of fermions and calculate the general non-renormalized effective potential depending on those, although this is not needed for scalar electrodynamics.
Let us begin with the scalar contribution to the effective potential. The scalars will be denoted by ϕ_a where the index a stands for the a-th scalar. For reasons of convenience we will sometimes refer to the vector $\boldsymbol{\phi}$ when talking about the whole collection of scalars, which is really only a notation. In fig. B.1 all relevant diagrams are shown. The propagators in the loops wear internal indices a and the external fields have vanishing momentum each. The vertices in fig. B.1 are given by

$$- iW_{ab}(\boldsymbol{\phi}) = -i\frac{\partial^2 V}{\partial\phi_a\partial\phi_b}, \tag{B.1}$$

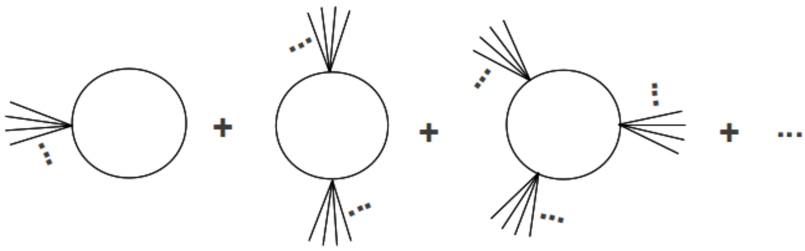


Figure B.1: Scalar one-loop contributions to the effective potential

which connects the two fields a and b, running in the loops, and the corresponding external fields.
The inner propagators contribute a factor

$$\frac{i}{k^2 + i\epsilon}, \tag{B.2}$$

where k is the momentum running in the loop which has to be integrated over.
We then have to sum over all internal field configurations, i.e. we have to multiply the matrices an then take the trace. Summing all diagrams then yields the scalar one-loop contribution to the effective potential

$$\begin{aligned} V_s &= i \int \frac{\mathrm{d}^4 k}{(2\pi)^4} \sum_{n=1}^{\infty} \frac{1}{2n} \operatorname{Tr}\left[\left(\frac{W(\phi)}{k^2 + i\epsilon}\right)\right]^n \\ &= \operatorname{Tr}\left[i \int \frac{\mathrm{d}^4 k}{(2\pi)^4} \sum_{n=1}^{\infty} \frac{1}{2n} \left(\frac{W}{k^2 + i\epsilon}\right)^n\right] \\ &= \operatorname{Tr} \frac{1}{2} \int \frac{\mathrm{d}^4 k}{(2\pi)^4} \ln\left(k^2 + W\right), \end{aligned} \tag{B.3}$$

where the 0-component of k has been Wick rotated into the complex plane like already done before.
This integral is again divergent. We choose an ultraviolet cut-off Λ to regularize. As the integrand depends only on the absolute value of the 4 dimensional euclidean vector k, we choose polar coordinates to evaluate the integral. The differential thus transforms like

$$\mathrm{d}^4 k \to \Omega_4 k; \tag{B.4}$$

where the Ω_4 denotes the area of the unit sphere in 4 dimensional euclidean space and the k on the right hand side represents the absolute value of the euclidean vector. The unit sphere in d dimensions is given by

$$\Omega_d = \frac{2\pi^{d/2}}{\Gamma(\frac{1}{2}d)}, \tag{B.5}$$

where Γ is the gamma function. Thus for 4 dimensions we obtain $\Omega_4 = 2\pi^2$ and we get

$$\begin{aligned}
\frac{1}{2}\int\frac{\mathrm{d}^4k}{(2\pi)^4}\ln\left(k^2+W\right) &= \frac{2\pi^2}{2(2\pi)^4}\int_0^\Lambda \mathrm{d}k\, k^3 \ln k^2 + W \\
&= \frac{1}{16\pi^2}\frac{1}{8}\left[-\Lambda^4 + 2\Lambda^2 W + 2W^2\ln W + 2(\Lambda^4 - W^2)\ln\left(\Lambda^2+W\right)\right] \\
&= \frac{1}{64\pi^2}\left[\Lambda^2 W + W^2\ln W + (\Lambda^4 - W^2)\left(\ln\left(1+\frac{W}{\Lambda^2}\right)+\ln\Lambda^2\right)\right] \\
&\approx \frac{1}{64\pi^2}\left[\Lambda^2 W + W^2\ln W + (\Lambda^4 - W^2)\left(\frac{W}{\Lambda^2}+\ln\Lambda^2\right)\right] \\
&= \frac{1}{64\pi^2}\left[\Lambda^2 W + W^2\ln W + \Lambda^2 W + \Lambda^4\ln\Lambda^2 - \frac{W^3}{\Lambda^2} - W^2\ln\Lambda^2\right] \\
&= \frac{1}{64\pi^2}\left[2\Lambda^2 W + W^2\ln\frac{W}{\Lambda^2}\right] \\
&= \frac{\Lambda^2}{32\pi^2}W + \frac{W^2}{64\pi^2}\ln\frac{W}{\Lambda^2}.
\end{aligned} \tag{B.6}$$

We neglected all infinite but irrelevant constants like $-\Lambda^4$ in line 2 and $\Lambda^4\ln\Lambda^2$ in line 5. Furthermore we took into account that Λ is supposed to be huge compared to all other scales and will later even be approached to infinity. Thus the factor $-\frac{W^3}{\Lambda^2}$ in line 5 was neglected afterwards. A further result of this consideration of large Λ is getting important in line 3 where the term $\ln(1+\frac{W}{\Lambda^2})$ was expanded to linear order around 1 and can thus be approached by $\frac{W}{\Lambda^2}$.
Our result for the general scalar contribution to the effective potential to one-loop order is now given by

$$V_s = \mathrm{Tr}\left[\frac{\Lambda^2}{32\pi^2}W(\phi) + \frac{W(\phi)^2}{64\pi^2}\ln\frac{W(\phi)}{\Lambda^2}\right]. \tag{B.7}$$

For individual classical potentials this has to be evaluated and renormalized by adding a finite number of counter terms. Assuming that we are treating renormalizable theories, we can neglect the terms depending on Λ indicating that we can absorb them in the counter terms. Not writing them down explicitly, but having in mind that they have to be taken into account when renormalizing, we can write the scalar one-loop potential in the rather abstract form

$$V_s = \mathrm{Tr}\left[\frac{W(\phi)^2}{64\pi^2}\ln W(\phi)\right]. \tag{B.8}$$

Again, it has to be emphasized that this expression does not represent a certain renormalization scheme but is an abstract expression to remember it more easily. Reinforcing the counter terms will also take care of the wrong dimensionality in the logarithm.
As the intention of this chapter is to derive the Coleman-Weinberg potential, we will now apply this result explicitly to massless scalar electrodynamics. Its Lagrangian reads

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + (D_\mu\phi)^* D^\mu\phi - \frac{\lambda}{6}\left|\phi\right|^4, \tag{B.9}$$

where $D_\mu = \partial_\mu + ieA_\mu$ and ϕ is a complex scalar field

$$\phi = \frac{1}{\sqrt{2}}(\phi_1 + i\phi_2). \tag{B.10}$$

The classical potential can thus be obtained from the Lagrangian

$$V = \frac{\lambda}{6}\,|\phi|^4 = \frac{1}{4!}(\phi_1 + \phi_2)^2. \tag{B.11}$$

Using this and the definition eq. (B.1), the matrix elements of $W(\boldsymbol{\phi})$ can be calculated. Thus

$$\begin{aligned} W_{11} &= \frac{\lambda}{6}(3\phi_1^2 + \phi_2^2) \\ W_{22} &= \frac{\lambda}{6}(3\phi_2^2 + \phi_1^2) \\ W_{12} = W_{21} &= \frac{\lambda}{3}\phi_1\phi_2. \end{aligned} \tag{B.12}$$

To make life easier, we diagonalize this matrix, i.e. we want to find unitary matrices such that $W = U^\dagger W_D U$, where W_D is a diagonal matrix. Writing W in this way, we find that

$$\ln W = U^\dagger \ln W_D U, \tag{B.13}$$

which becomes apparent when writing the logarithm in its expansion series. It then becomes clear that we can write

$$\mathrm{Tr}\left[\frac{W^2}{64\pi^2}\ln W\right] = \mathrm{Tr}\left[\frac{W_D^2}{64\pi^2}\ln W_D\right] = \sum_i \frac{\alpha_i^2}{64\pi^2}\ln\alpha_i, \tag{B.14}$$

where the α_i are the eigenvalues of W. We find

$$\alpha_1 = \lambda\,|\phi|^2 \quad \text{and} \quad \alpha_2 = \frac{1}{3}\lambda\,|\phi|^2\,. \tag{B.15}$$

As a consequence we get

$$\begin{aligned} V_s &= \frac{1}{64\pi^2}\left[\lambda^2\,|\phi|^4 \ln\left(\lambda\,|\phi|^2\right) + \frac{\lambda^2}{9}\,|\phi|^4 \ln\left(\frac{1}{3}\,|\phi|^2\right)\right] \\ &= \frac{1}{64\pi^2}\left[\lambda^2\,|\phi|^4\left(\frac{10}{9}\ln\left(\lambda\,|\phi|^2\right) - \frac{1}{9}\ln 3\right)\right] \\ &= \frac{1}{16\pi^2}\left[\frac{5\lambda^2}{18}\,|\phi|^4\left(\ln\left(\lambda\,|\phi|^2\right) - \frac{1}{10}\ln 3\right)\right] \\ &= \frac{1}{16\pi^2}\left[\frac{5\lambda^2}{18}\,|\phi|^4\left(\ln\left(\lambda\,|\phi|^2\right)\right) - \delta_1\right], \end{aligned} \tag{B.16}$$

where $\delta_1 = \frac{1}{10}\ln 3$. This is the non-renormalized scalar one-loop contribution to the effective potential. The next step will be to calculate the vector boson contribution.

To do so let us first get back to the most general case in order to find a solution beyond the special case of massless scalar electrodynamics. Like before we define a matrix in the following way

$$\mathcal{L} = ... - \frac{1}{2} \sum_{a,b} M_{ab}^2(\boldsymbol{\phi}) A_{\mu a} A_B^\mu + ..., \tag{B.17}$$

which is quadratic in $\boldsymbol{\phi}$ and describes the interactions between scalars and vector bosons. The $A_{\mu a}$ are the vector fields, where the greek indices are like always the Lorentz indices and the a's refer to the numeration of the fields. The diagrams of interest are basically those from fig. B.1 except for the fact that the propagators in the loops are not scalar but vector propagators. Beyond that the vertices can be read off the Lagrangian and are given by

$$iM^2 g_{\mu\nu}. \tag{B.18}$$

The vector propagators in Landau gauge contribute a factor

$$-i \frac{g_{\mu\nu} - k_\mu k_\nu / k^2}{k^2 + i\epsilon}. \tag{B.19}$$

With the new vertices and propagators and the argumentation from before, we get the vector one-loop contribution to the effective potential

$$V_v = i \int \frac{\mathrm{d}^4 k}{(2\pi)^4} \sum_{n=1}^{\infty} \frac{3}{2n} \operatorname{Tr} \left[\left(\frac{M^2(\boldsymbol{\phi})}{k^2 + i\epsilon} \right)^n \right], \tag{B.20}$$

where the factor of 3 emerges when contracting the vertices with the propagators. Performing the same calculation like before, we find

$$V_v = 3 \operatorname{Tr} \left[\frac{\Lambda^2}{32\pi^2} M(\boldsymbol{\phi})^2 + \frac{M(\boldsymbol{\phi})^4}{64\pi^2} \ln \frac{M(\boldsymbol{\phi})^2}{\Lambda} \right] \tag{B.21}$$

and in the more abstract form

$$V_v = \frac{3}{64\pi^2} \operatorname{Tr} \left[M(\boldsymbol{\phi})^4 \ln M(\boldsymbol{\phi})^2 \right]. \tag{B.22}$$

Equipped with this finding we can go back to massless scalar electrodynamics and apply it. There we have only one vector boson, let us call it photon, and our matrix M is only a number, which reads

$$M^2 = 2e^2 \left|\phi\right|^2 = e^2(\phi_1^2 + \phi_2^2). \tag{B.23}$$

Plugging this into the general formula, yields

$$\begin{aligned} V_v &= \frac{3}{64\pi^2} \left[4e^4 \left|\phi\right|^4 \ln \left(2e^2 \left|\phi\right|^2 \right) \right] \\ &= \frac{3}{16\pi^2} \left[e^4 \left|\phi\right|^4 \left(\ln \left(e^2 \left|\phi\right|^2 \right) \right) - \delta_2 \right], \end{aligned} \tag{B.24}$$

where $\delta_2 = -\ln 2$.

Now summing the tree-level potential and the scalar and vector one-loop contribution to the effective potential, yields

$$V_{\text{eff}} = \frac{\lambda}{6}|\phi|^4 + \frac{1}{16\pi^2}\left(\frac{5\lambda^2}{18} + 3e^4\right)|\phi|^4 \ln|\phi|^2 - \left(\delta_1\lambda^2 + \delta_2 e^4\right)|\phi|^4 \,, \tag{B.25}$$

where δ_1 and δ_2 were redefined accordingly. We will see later that the exact form of these factors is not important. To make it complete we reenact the counter terms

$$V_{\text{eff}} = \frac{\lambda}{6}|\phi|^4 + \frac{1}{16\pi^2}\left(\frac{5\lambda^2}{18} + 3e^4\right)|\phi|^4 \ln|\phi|^2 - \left(\delta_1\lambda^2 + \delta_2 e^4\right)|\phi|^4 + B|\phi|^2 + C|\phi|^4 \,. \tag{B.26}$$

We are now in the position to impose renormalization conditions. Like before we renormalize according to the on-shell scheme. I.e.

$$\left.\frac{d^2 V_{\text{eff}}}{d|\phi|^2}\right|_{|\phi|=0} \stackrel{!}{=} 0 \tag{B.27}$$

and

$$\left.\frac{d^4 V_{\text{eff}}}{d|\phi|^4}\right|_{|\phi|=M} \stackrel{!}{=} 4\lambda, \tag{B.28}$$

where the factor 4 comes from the normalization of ϕ.

The first condition yields

$$\begin{aligned} \left.\frac{d^2 V_{\text{eff}}}{d|\phi|^2}\right|_{|\phi|=0} &= 2B \stackrel{!}{=} 0 \\ \Rightarrow B &= 0. \end{aligned} \tag{B.29}$$

The second condition yields

$$\begin{aligned} \left.\frac{d^4 V_{\text{eff}}}{d|\phi|^4}\right|_{|\phi|=M} &= 4\lambda - 4!(\delta_1\lambda^2 + \delta_2 e^4) + 4!C + \frac{1}{16\pi^2}\left(\frac{5\lambda^2}{18} + 3e^4\right) 4!\left(\ln M^2 + \frac{25}{6}\right) \\ &\stackrel{!}{=} 4\lambda \\ \Rightarrow C =& (\delta_1\lambda^2 + \delta_2 e^4) - \frac{1}{16\pi^2}\left(\frac{5\lambda^2}{18} + 3e^4\right)|\phi|^4 \ln\left(\ln\frac{|\phi|^2}{M^2} - \frac{25}{6}\right). \end{aligned} \tag{B.30}$$

Finally we arrive at the result, the Coleman-Weinberg potential

$$V_{\text{eff}} = \frac{\lambda}{6}|\phi|^4 + \frac{1}{16\pi^2}\left(\frac{5\lambda^2}{18} + 3e^4\right)|\phi|^4\left(\ln\frac{|\phi|^2}{M^2} - \frac{25}{6}\right). \tag{B.31}$$

As long as $\lambda \propto e^4$ one can ensure that $\ln \frac{|\phi|^2}{M^2}$ does not have to be large and the loop expansion is valid. The conceptual difference to simple scalar field theory is that in scalar field theory the tree-level potential can be cancelled by the photon one-loop potential to reach the minimum. That the condition $\lambda \propto e^4$ can indeed be realized can be seen by a renormalization group analysis of the parameters λ and e.

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Erklärung:

Ich versichere, dass ich diese Arbeit selbstständig verfasst habe und keine anderen als die angegebenen Quellen und Hilfsmittel benutzt habe.

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